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Key Points:

- The phase lag effect on groundwater flow is considered in the analytical model
- Sensitivity analysis is performed based on the solution of lagging flow model
- The aquifer parameters are analyzed for field observed data based on the model

Supporting Information:

- Supporting Information S1
- Data Set S1

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A Lagging Model for Describing Drawdown Induced by a Constant-Rate Pumping in a Leaky Confined Aquifer

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Abstract This study proposes a generalized Darcy's law with considering phase lags in both the water flux and drawdown gradient to develop a lagging flow model for describing drawdown induced by constant-rate pumping (CRP) in a leaky confined aquifer. The present model has a mathematical formulation similar to the dual-porosity model. The Laplace-domain solution of the model with the effect of wellbore storage is derived by the Laplace transform method. The time-domain solution for the case of neglecting the wellbore storage and well radius is developed by the use of Laplace transform and Weber transform. The results of sensitivity analysis based on the solution indicate that the drawdown is very sensitive to the change in each of the transmissivity and storativity. Also, a study for the lagging effect on the drawdown indicates that its influence is significant associated with the lag times. The present solution is also employed to analyze a data set taken from a CRP test conducted in a fractured aquifer in South Dakota, USA. The results show the prediction of this new solution with considering the phase lags has very good fit to the field data, especially at early pumping time. In addition, the phase lags seem to have a scale effect as indicated in the results. In other words, the lagging behavior is positively correlated with the observed distance in the Madison aquifer.

1. Introduction

The constant-rate pumping (CRP) test is one of aquifer tests to characterize the hydraulic properties (e.g., transmissivity and storativity) of aquifers. For the CRP, the test well is pumped at a constant flowrate and temporal drawdown data are recorded at the observation wells. The hydraulic parameters can then be determined either by an analytical drawdown solution coupled with an optimization method or simply by a type-curve method when analyzing the observed data. A variety of studies describing the transient drawdown behavior induced by the CRP in confined aquifers had been presented (e.g., Hantush, 1964; Lin et al., 2016; Papadopulos & Cooper, 1967; Theis, 1935; Yeh & Chang, 2013). Theis (1935) was the first to propose a mathematical model for drawdown induced by a CRP in a confined aquifer of infinite extent based upon analogy with transient heat conduction problem. He treated the pumping well as a line source implying that the pumping well has zero radius. Hantush (1964, p. 340) improved Theis (1935) model with considering a finite well radius. As regard to large-diameter well, the effect of wellbore storage on the drawdown cannot be ignored. Papadopoulos and Cooper (1967) therefore proposed an analytical model for a CRP by adding a term representing the wellbore storage effect in the inner boundary condition. Conducting a CRP test, a well skin may be developed near the well as a result of well drilling and/or development. A skin factor is commonly introduced to represent the effect of skin zone on the pumping drawdown. For the development of the solution with considering the skin effect, there are many approaches available in the literature (e.g., Agarwal et al., 1970; Cassiani et al., 1999; Hawkins, 1956; Hurst, 1953; Moench, 1997; Moench & Hsien, 1985; Sandal et al., 1978; van Everdingen, 1953). If an aquifer is bounded by a semipermeable layer (also called as aquitard), the groundwater would recharge to the pumped aquifer from the adjacent aquifer through the aquitard. The recharge is referred to as leakage. Numerous models have been developed for describing the flow in a leaky aquifer system (e.g., Hantush, 1960; Hantush & Jacob, 1955; Lin et al., 2017; Malama et al., 2007; Neuman & Witherspoon, 1969; Zlotnik & Zhan, 2005). Hantush and Jacob (1955) developed two different forms of analytical solution for describing the spatiotemporal drawdown distribution induced by a constant-rate pumping in an infinite leaky aquifer with an initially uniform head. The flow, or leakage, into the aquifer is considered vertical and proportional to the hydraulic gradient between the aquifer and adjacent aquifer. The leakage is treated as a source/sink term embedded in the governing equation they solved. Because of its simplicity, their solution is widely adopted to predict the pumping drawdown or estimate the aquifer parameters in a leaky confined aquifer system.

The abovementioned models are mostly based on the classical groundwater flow equation (i.e., diffusion-type equation) derived from Darcy's law and mass conservation. In reality, Darcy's law describes that the water flow occurs instantaneously once the drawdown gradient exits, implying that the drawdown change can be measured immediately at the observation wells anywhere in the aquifer as water is pumped at the test well. Strictly speaking, this assumption may not really true because the propagation speed of drawdown in the media might not be infinite. Darcy's law is derived based on the momentum equation without considering the inertial and viscous forces (Viessman & Lewis, 1996, p. 441). Some studies have been reported to address the issue of inertial effect on the groundwater flow (e.g., Löfqvist & Rehbinder, 1993; Pascal, 1986). Löfqvist and Rehbinder (1993) proposed a groundwater flow equation with considering the effect of inertial forces on the fluid. The equation they obtained is indeed a wave equation implying that the pressure propagation has a finite speed. They found that the inertial effect is important when the operating time is smaller than the relaxation time defined as the product of fluid density and permeability divided by the product of viscosity and aquifer porosity. They further indicated that the change of the well water level would cause the oscillation due to the finite propagation velocity of the water in the aquifer and the inertia of water in the well.

The void space in porous media may be classified into two kinds: interconnected pores and noninterconnected pores. The groundwater fluid can move from one pore to nearby connected pore, which is closely related to the definition of effective porosity. The noninterconnected pores, on the other hand, include the dead-end pores and very small pores in which the water fluid is practically immobile (Bear & Cheng, 2010, p. 74). Fatt (1959) conducted laboratory experiments and indicated that the transient pressure distribution in porous media might be influenced by the presence of dead-end pores. He mentioned that there may be a pressure difference between the dead-end pore and adjacent flow channels. The dead-end pore will act as a source due to the pressure difference and the flow equation then should include a source function. Instead of doing that, we propose a lagging model to simulate the flow in groundwater systems subject to the effect of noninterconnected pores. It has been known from field observations that the temporal drawdown distribution induced by pumping in a fractured aquifer may exhibit a flattening portion in the middle pumping period (e.g., Cho et al., 2004; Greene et al., 1999; Moench, 1984; Najurieta, 1980; Tiedeman & Hsieh, 2001). The phenomenon of having this near flat portion may be referred to as the delayed response. Such a response is explained based on the concept of dual-porosity model due to the exchange of water between the fracture and matrix (Belcher et al., 2001). In the dual-porosity model, the fracture network of the aquifer is considered to be well interconnected; however, natural fractures are not always well connected in reality and some of them may be isolated with finite lengths (Krásný & Sharp, 2007, p. 474). Similar to the noninterconnected pores in the porous media, the noninterconnected fractures in fractured aquifers would lead to extra time for establishing the flow field. The groundwater flow in a fractured aquifer may therefore exhibit a lagging behavior.

Fourier's law in the heat conduction area, analogous to Darcy's law in groundwater flow area, is an empirical formula describing the linear relationship between the heat flux and the temperature gradient in solids. The law implies that the speed of heat propagation in the solid is infinite. Tzou (2014) mentioned that the heat propagation was found with finite speeds based on experimental observations of ultrafast laser heating on gold films. The concerns on the heat transfer equation on the basis of classical Fourier's law have been raised when the equation is used to describe the ultrafast process of heat transport. In the past, Cattaneo (1958) and Vernotte (1958, 1961) assumed that the heat flux and thermal gradient does not occur at the same time and allowed that the temperature gradient can precede the heat flux vector in the law. In other words, there will be a delayed response in the heat flux in contrast to the thermal gradient. Their approaches are referred to as CV model and their assumption leads the heat propagation from diffusion-like process to wave-like process. The former process for the heat equation implies that the speed of heat propagation is infinite but the latter for the wave equation indicates that the speed of heat propagation is finite (see e.g., Baumeister & Hamill, 1969; Vick & Özisk, 1983). Tzou (1995) extended CV model and allowed the heat flux to precede the temperature gradient based on a concept that the thermal lagging response in time can be comparable to the characteristic time to the phase lag characterizing the microstructural interactions in conducting materials. Tzou's (1995) approach is known as dual-phase lag model, which can be considered as a generalized Fourier's law for heat conduction problems. The model allows temperature gradient to precede the heat flux or the reverse. In other words, the lag time may occur in heat flux, thermal gradient, or both.

In analogy to the concept of Tzou (1995), this study presents an analytical lagging model for predict drawdown induced by a CRP in a leaky confined aquifer based on a generalized Darcy's law accounting for the inertial effect and the effect of noninterconnected pores or fractures. The model also takes account of the effects of wellbore storage, wellbore skin, and leakage from the aquitard. The term representing the leakage effect is adopted from Hantush and Jacob's (1955) work. The semianalytical solution for the model is developed by the Laplace transform technique. The transient analytical solution without wellbore storage effect is also provided. On the basis of present solution, the lagging effect in the water flux and drawdown gradient are discussed. Additionally, a sensitivity analysis based on the present solution is performed to investigate the drawdown behavior in response to the change in each of parameters of the aquifer system. The present solution is applied to estimate the aquifer parameters when analyzing a set of field measured drawdown data obtained from a CRP test conducted in South Dakota, USA.

2. Methodology

The classical theory for groundwater equation based on Darcy's law implies that the water flux q occurs immediately once the drawdown gradient ∇s is produced. The Darcy' s law in radial direction can be expressed as:

$$q_r(r,t) = -K\frac{\partial s(r,t)}{\partial r} \tag{1}$$

where r is the radial distance from the center of the pumping well, t is the operating time, and K is the hydraulic conductivity. Similar to Fourier's law, Baumeister and Hamill (1969, p. 543) stated that many investigators have pointed out that the propagation velocity of heat resulting from the Fourier heat-conduction equation is infinite. Vick and Özisk (1983, p. 902) mentioned that temperature disturbances will propagate at infinite velocities according to classical heat conduction theory. Beck et al. (1992, p. 41) also mentioned that heat introduced at one point in a solid body will be instantaneously transmitted to the entire body as predicted by Fourier's law of conduction; therefore, equation (1) indicates that the drawdown propagation has an infinite speed implying that the drawdown can be measured immediately at any observation wells once the aquifer is under pumping. However, the propagation speed of drawdown in aquifers may be finite and the assumption of Darcy's law may be unrealistic in some real-world problems. Based on Tzou's (1995) dual-phase-lag concept, a generalized Darcy's law assuming that the q_r and ∇s occurring at different times is denoted as

$$q_r\left(r,t+\tau_q\right) = -K\frac{\partial s(r,t+\tau_s)}{\partial r} \tag{2}$$

where τ_q and τ_s are respectively the phase lags of the water flux and drawdown gradient. In analogy to Tzou's (1995) approach, the τ_q is introduced to reflect the fast transient effect of water flow inertia because of the moderately high velocity of water flow in a high permeability medium or large opening in aquifers. The inertial effect causes the energy passing the water and moving like waves; consequently, the drawdown propagation arrives later than expected. The τ_s is considered induced by the effect of structural interactions due to noninterconnected pore in porous aquifers or noninterconnected fractures in fractured media leading to an extra time for establishing the drawdown gradient field. In reality, both τ_q and τ_s may range from few seconds to several hours. Note that equation (2) is used to describe local behavior in the confined aquifer. If the flow system has a Neumann-type boundary with a given flux value, the drawdown gradient may precede the water flux. In regional flow systems, the boundary condition does not ensure the priority of the water flux and drawdown gradient.

Applying the Taylor series expansion to equation (2), the first-order approximation for generalized Darcy's law then results in

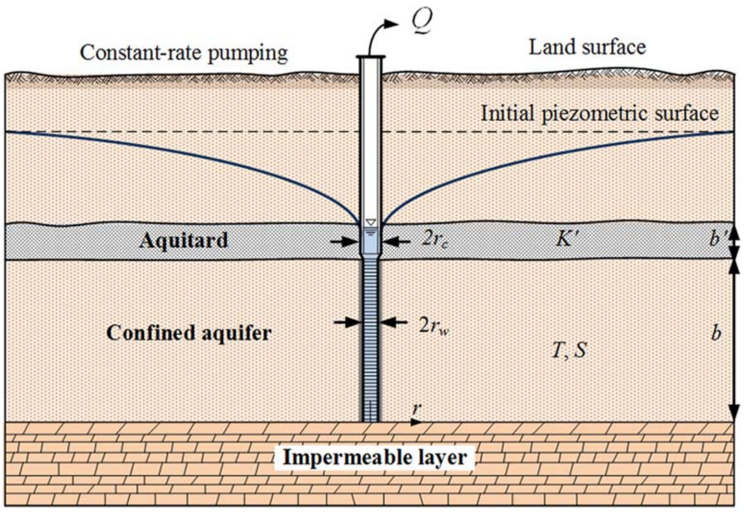

Figure 1. Schematic representation for a constant-rate pumping conducted at a fully penetrating well in a leaky confined aquifer.

$$q_r(r,t)+\tau_q\frac{\partial q_r(r,t)}{\partial t}\cong -K\left(\frac{\partial s(r,t)}{\partial r}+\tau_s\frac{\partial^2 s(r,t)}{\partial t\partial r}\right) \tag{3}$$

2.1. Mathematical Model

Figure 1 shows the schematic representation for the mathematical model describing the drawdown due to CRP at a fully penetrating well with finite radius, r_w, and casing radius, r_c, in a leaky confined aquifer of infinite lateral extent with uniform thickness b and b' for aquifer and aquitard, respectively. The effect of leakage from aquitard treated in Hantush and Jacob (1955) is adopted in this model. The effects of wellbore storage and wellbore skin on the drawdown are also considered in the model. The origin of the coordinate is located at the center of the well and at the bottom of the main aquifer.

Substituting equation (3) into mass conservation equation,

$$-\frac{1}{r}\frac{\partial}{\partial r}(rq_r)=S\frac{\partial s}{\partial t}+\frac{K'}{B'}s \tag{4}$$

the groundwater equation can then be obtained as

$$\left(1+\tau_s\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)-\left(1+\tau_q\frac{\partial}{\partial t}\right)\frac{K's}{B'T}=\frac{S}{T}\left(1+\tau_q\frac{\partial}{\partial t}\right)\frac{\partial s}{\partial t} \tag{5}$$

where K' is the is the hydraulic conductivity of the aquitard, B' is the thickness of the aquitard, T and S are respectively the transmissivity (i.e., $K\times b$) and storativity of the aquifer. Note that equation (5) is a wave equation while the groundwater flow equation is a diffusion equation. Before the pumping, the drawdown in the aquifer is considered to be at stationary state; hence, the initial conditions are denoted as

$$s(r,0)=\frac{\partial s(r,0)}{\partial t}=0 \tag{6}$$

The flowrate across the well screen is maintained at a constant value Q and the effect of the wellbore storage is included (Papadopoulos & Cooper, 1967); thus, the inner boundary condition for the model can be written as:

$$2\pi r_w T\frac{\partial s(r_w,t)}{\partial r}=-Q+\pi r_c^2\frac{\partial H(t)}{\partial t} \tag{7}$$

where $H(t)$ is the drawdown (or water level) in the pumping well. Considering the well skin effect on the flow, the condition for the drawdown at the wellbore can be expressed as

$$H(t)=s(r_w,t)-S_k r_w\frac{\partial s(r_w,t)}{\partial r} \tag{8}$$

where S_k is a skin factor. At the remote boundary (i.e., $r\to\infty$), the drawdown is assumed to be zero; therefore, it can be expressed as

$$\lim_{r\to\infty} s(r,t)=0 \tag{9}$$

Define the following dimensionless variables and parameters:

$$s_D=\frac{4\pi T}{Q}s,\ H_D=\frac{4\pi T}{Q}H,\ r_D=\frac{r}{r_w},\ t_D=\frac{T}{Sr_w^2}t,\ C_D=\frac{r_c^2}{2Sr_w^2},\ \tau_{qD}=\frac{T}{Sr_w^2}\tau_q,\ \tau_{sD}=\frac{T}{Sr_w^2}\tau_s,\ \kappa=\frac{K'}{B'T} \tag{10}$$

On the basis of equation (10), the dimensionless equations for equations (5)–(9) can be respectively written as

$$\left(1+\tau_{sD}\frac{\partial}{\partial t_D}\right)\left(\frac{\partial^2 s_D}{\partial r_D^2}+\frac{1}{r_D}\frac{\partial s_D}{\partial r_D}\right)-\left(1+\tau_{qD}\frac{\partial}{\partial t_D}\right)\kappa s_D=\left(1+\tau_{qD}\frac{\partial}{\partial t_D}\right)\frac{\partial s_D}{\partial t_D} \tag{11}$$

$$s_D(r_D,0)=\frac{\partial s_D(r_D,0)}{\partial t_D}=0 \tag{12}$$

$$\frac{\partial s_D(1,t_D)}{\partial r_D}=-2+C_D\frac{\partial H_D(t_D)}{\partial t_D} \tag{13}$$

$$H_D(t_D)=s_D(1,t_D)-S_k\frac{\partial s_D(1,t_D)}{\partial r_D} \tag{14}$$

and

$$\lim_{r_D\to\infty} s_D(r_D,t_D)=0 \tag{15}$$

2.2. The Present Solution

Applying the Laplace transform to equation (11) yields

$$(1+\tau_{sD}p)\left(\frac{\partial^2\bar{s}_D}{\partial r_D^2}+\frac{1}{r_D}\frac{\partial\bar{s}_D}{\partial r_D}\right)-(1+\tau_{qD}p)\kappa\bar{s}_D=(1+\tau_{qD}p)p\bar{s}_D \tag{16}$$

where the overbar represents the term in Laplace domain and p is the Laplace variable. The transformed boundary conditions for equations (13)–(15) are respectively

$$\frac{\partial\bar{s}_D(1,t_D)}{\partial r_D}=-\frac{2}{p}+C_Dp\bar{H}_D(p) \tag{17}$$

$$p\bar{H}_D(p)=\bar{s}_D(1,p)-S_k\frac{\partial\bar{s}_D(1,p)}{\partial r_D} \tag{18}$$

and

$$\lim_{r_D\to\infty}\bar{s}_D(r_D,p)=0 \tag{19}$$

The general solution of equation (16) can be obtained as

$$\bar{s}_D=AI_0(\lambda r_D)+BK_0(\lambda r_D) \tag{20}$$

where $\lambda=\sqrt{(p+\kappa)(1+p\tau_{qD})}/\sqrt{1+p\tau_{sD}}$, A and B are undetermined coefficients, and I_0 and K_0 are respectively the modified Bessel function of first kind and second kind of order zero. Imposing the boundary conditions equations (17–19) on equation (20), A and B can then be respectively obtained as

$$A=0 \tag{21}$$

and

$$B=\frac{2}{C_Dp^2K_0(\lambda)+\lambda p(1+C_DpS_k)K_1(\lambda)} \tag{22}$$

Finally, the dimensionless drawdown solution in Laplace domain can be expressed as

$$\bar{s}_D=\frac{2K_0(\lambda r_D)}{C_Dp^2K_0(\lambda)+\lambda p(1+C_DpS_k)K_1(\lambda)} \tag{23}$$

where K_1 is the modified Bessel function of second kind of order one.

Neglecting the wellbore storage effect on the drawdown, the dimensionless analytical solution obtained by applying Laplace transform and Weber transform is given as follows and its derivation is shown in Appendix A.

$$s_D=\frac{2}{\pi}\int_0^\infty\frac{\left\{\beta_2+\beta_2e^{\frac{\beta_2t_D}{\tau_{qD}}}-2\beta_2e^{\frac{(1+\beta_2+\kappa\tau_{qD}+a^2\tau_{sD})t_D}{2\tau_{qD}}}+\left(e^{\frac{\beta_2t_D}{\tau_{qD}}}-1\right)\left[1+\kappa\tau_{qD}-(a^2+2\kappa)\tau_{sD}\right]\right\}}{(a^2+\kappa)\beta_2}\times e^{-\frac{(1+\beta_2+\kappa\tau_{qD}+a^2\tau_{sD})t_D}{2\tau_{qD}}}\frac{J_0(ar_D)Y_1(a)-Y_0(ar_D)J_1(a)}{J_1^2(a)+Y_1^2(a)}da \tag{24}$$

with

$$\beta_1=\sqrt{1-4(a^2+\kappa)\tau_{qD}+2a^2\tau_{sD}+a^4{\tau_{sD}}^2} \tag{25}$$

and

$$\beta_2=\sqrt{\left(1+\kappa\tau_{qD}+a^2\tau_{sD}\right)^2-4(a^2+\kappa)} \tag{26}$$

where J_0 and Y_0 are respectively the Bessel function of first kind and second kind of order zero; J_1 and Y_1 are respectively the Bessel function of first kind and second kind of order one.

It is noteworthy that if $\tau_{qD}=\tau_{sD}=0$, equation (24) can be written as

$$s_D=\frac{2}{\pi}\int_0^\infty \frac{2-2e^{-(\kappa+a^2)t_D}}{(a^2+\kappa)}\frac{J_0(ar_D)Y_1(a)-Y_0(ar_D)J_1(a)}{J_1^2(a)+Y_1^2(a)}da \tag{27}$$

If considering the pumping well of infinitesimal radius and by letting $r_w\rightarrow 0$, one can obtain following relationship

$$\lim_{\gamma\rightarrow 0}\frac{J_0(ar_D)Y_1(a\gamma)-Y_0(ar_D)J_1(a\gamma)}{\gamma(J_1^2(a\gamma)+Y_1^2(a\gamma))}=-\frac{a\pi}{2}J_0(ar_D) \tag{28}$$

Then, equation (27) can reduce to Hantush and Jacob's (1955, p. 96) solution denoted as

$$s_D=2\int_0^\infty \frac{1-e^{-(\kappa+a^2)t_D}}{a^2+\kappa}aJ_0(ar_D)da \tag{29}$$

2.3. Comparison of Present Model With Dual-Porosity Model

For the dual-porosity model proposed by Barenblatt et al. (1960), the drawdown distributions in fracture and matrix induced by pumping can be respectively written as

$$T\left(\frac{\partial^2 s_f}{\partial r^2}+\frac{1}{r}\frac{\partial s_f}{\partial r}\right)=S_f\frac{\partial s_f}{\partial t}+\beta(s_f-s_m) \tag{30}$$

and

$$S_m\frac{\partial s_m}{\partial t}=\beta(s_f-s_m) \tag{31}$$

where the subscripts f and m represent in the fracture and the matrix, respectively, and β is the rate of water exchange between the fracture and the matrix. Substituting equation (30) into equation (31), the drawdown in the fracture can then be written as

$$\left(1+\frac{S_m}{\beta}\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 s_f}{\partial r^2}+\frac{1}{r}\frac{\partial s_f}{\partial r}\right)=\frac{S_f+S_m}{T}\left(1+\frac{S_fS_m}{\beta(S_f+S_m)}\frac{\partial}{\partial t}\right)\frac{\partial s_f}{\partial t} \tag{32}$$

Equation (5) without the leakage effect can be expressed as

$$\left(1+\tau_s\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)=\frac{S}{T}\left(1+\tau_q\frac{\partial}{\partial t}\right)\frac{\partial s}{\partial t} \tag{33}$$

Comparing equations (32) and (33), one may have:

$$\tau_s=S_m/\beta \tag{34}$$

$$S=S_f+S_m \tag{35}$$

and

$$\tau_q=S_mS_f/(\beta S_f+\beta S_m) \tag{36}$$

The present model is based on the concept of lagging response in Darcy's law while the dual-porosity model focuses on the water flow exchange between the fracture and the matrix. Interestingly, the governing equations of both models (i.e., equations (32) and (33)) are similar in mathematical forms, implying that the present model has some physical similarity to the dual-porosity model. The present lagging model allows the case of structural interaction (i.e., $\tau_q < \tau_s$) or inertial force (i.e., $\tau_q > \tau_s$) influencing the flow field . On the contrary, the dual-porosity model have to be subject to the constraint of $S_m S_f/(\beta S_f + \beta S_m) < S_m/\beta$ (i.e., $S_f/(S_f + S_m) < 1$) due to the reality that the parameters β, S_m, and S_f are all positive values. Consequently, the dual-porosity model cannot be use to depict the drawdown in a fractured aquifer system when the effect of inertia force dominates the flow. This is the limitation of the dual-porosity model as compared with the present model in the analysis of fractured flow.

3. Results and Discussion

3.1. The Effects of τ_{qD} and τ_{sD}

A pumping test is considered to conduct at a confined aquifer with the parameters T, S, K', B', and S_k being equal to 10 m^2/h, 0.0001 m^{-1}, 0 m/h, 0 m, and 0, respectively. The pumping well radius and casing radius are both 0.2 m with a constant rate $Q = 100$ m^3/h and the observation distance is located 10 m (or $r_D = 50$) from the pumping well. Figure 2 shows the temporal distribution curves of the dimensionless drawdown s_D for (a) $\tau_{sD} = 10^5$ (2.4 min), 10^6 (24 min), and 10^7 (4 h) with $\tau_{qD} = 10^3$, 10^4, and 10^5 and (b) $\tau_{qD} = 10^5$, 10^6, and 10^7 with $\tau_{sD} = 10^3$, 10^4, and 10^5. Figure 2a shows the cases that the structural interaction effect is dominant in the flow (i.e., $\tau_{qD} < \tau_{sD}$). For a given value of τ_{sD}, the pumping drawdown increases with dimensionless time, tends to be stabilized at the time near τ_{qD}, and increases again after the time close to τ_{sD}. The figure shows that a smaller τ_{qD} will have a larger drawdown before the dimensionless time approaching τ_{qD} . On the other hand, for a fixed value of τ_{qD}, a larger τ_{sD} would lead to a larger drawdown before the dimensionless time near τ_{sD}. It can be observed that the drawdown curves are stabilized in the period for the dimensionless time approximately from τ_{qD} to τ_{sD} and all the drawdown curves match together after the time close to τ_{sD}. Figure 2b demonstrates the cases that the inertial effect is dominant (i.e., $\tau_{sD} < \tau_{qD}$). For a fixed τ_{sD}, a larger τ_{qD} produces a smaller drawdown before the dimensionless time approaching τ_{qD}. Oppositely, a larger τ_{sD} results in a larger drawdown before the time near τ_{sD} for a fixed τ_{qD} and then all the drawdown curves merge after the time τ_{sD}. Note that the oscillatory phenomenon occurring during the dimensionless time from 10^4 to 10^5 can be attributed to the large phase lag influencing the water flow (i.e., $\tau_{qD} = 10^6$ and 10^7) because the drawdown propagation behaves more like wave for large τ_{qD}. Moreover, the lagging effect is vanished when $\tau_{sD} = \tau_{qD}$ no matter what values they are.

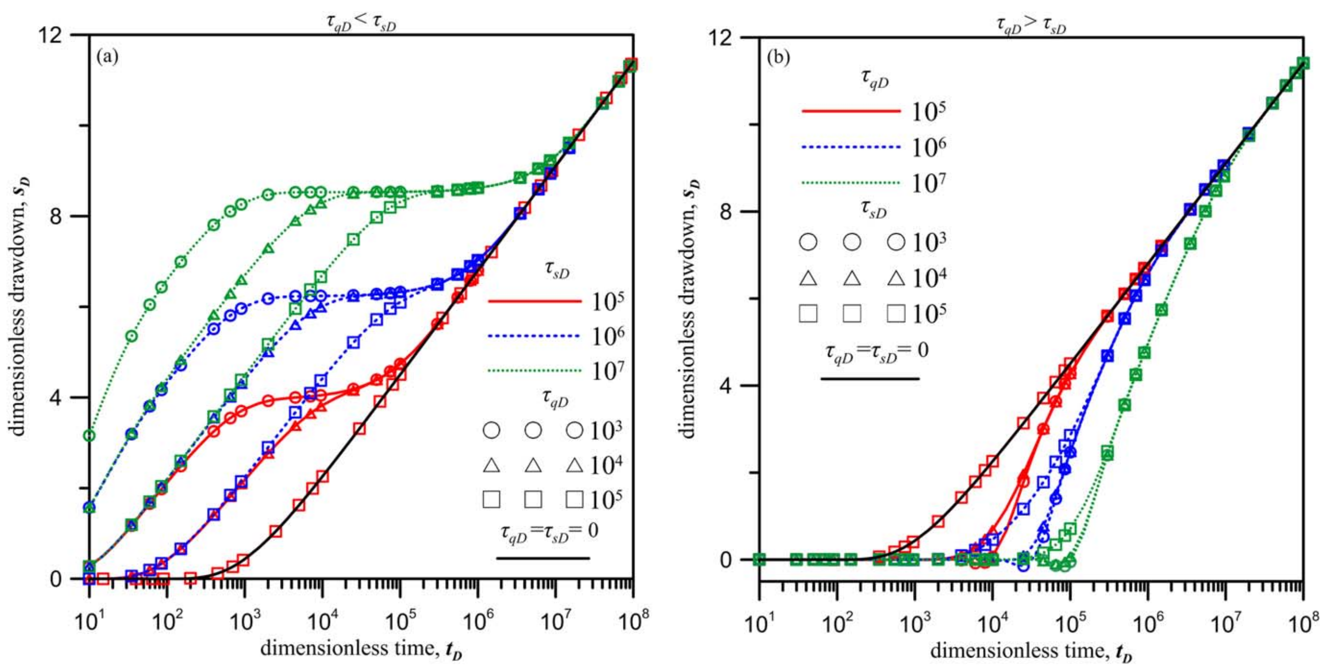

Figure 2. Temporal dimensionless drawdown distributions s_D for (a) τ_{sD} from 10^5 to 10^7 with τ_{qD} from 10^3 to 10^5 and (b) τ_{qD} from 10^5 to 10^7 with τ_{sD} from 10^3 to 10^5.

Table 1
Default Parameter Values in Sensitivity Analysis

Parameter	Value	Parameter	Value
Q	400 m^3/h	r_w	0.2 m
T	10 m^2/h	r_c	0.2 m
S	0.0001 m^{-1}	τ_q	0.2 h
K'	0.0001 m/h	τ_s	2 h
B'	50 m	S_k	0

3.2. Sensitivity Analysis

Sensitivity analysis is a useful tool in investigating the effect of uncertainty in each of aquifer and aquitard parameters on the aquifer drawdown. The normalized sensitivity is defined as (Kabala, 2001):

$$X_{i,k}=P_k \frac{\partial O_i}{\partial P_k} \tag{37}$$

where $X_{i,k}$ is the normalized sensitivity of the i^{th} output value O_i with respect to the k^{th} input parameter P_k. The term on the right-hand side of equation (37) can be approximated as:

$$\frac{\partial O_i}{\partial P_k}=\frac{O_i(P_k+\Delta P_k)-O_i(P_k)}{\partial P_k} \tag{38}$$

where ΔP_k may be approximated by $\Delta P_k=10^{-3}\ P_k$ (Huang & Yeh, 2007). In the following case study, the default parameter values are provided in Table 1.

Figure 3 demonstrates the temporal distribution curves of normalized sensitivity for drawdown in response to the change in each of parameters T, S, K', τ_q and τ_s. The sensitivity curves shown in the figure observed at the $r=10$ m indicate that the drawdown is sensitive to the changes in τ_q, τ_s, and S at early times before $t=0.03$ h (1.8 min) and the most sensitive to the change in T after that time. The parameter S is the second in the period 0.1 h $< t <$ 35 h while K' becomes the second after $t=35$ h. It can be observed that each of τ_q and τ_s significantly affects the drawdown in the period related to the value of the lag time. In addition, the effect of τ_q on the drawdown vanishes at $t=0.4$ h while the effect of τ_s disappears at $t=4$ h. The figure also displays that small positive perturbations in the values of T, S, τ_q, and K' result in negative influences on the drawdown. In contrast, a positive change in τ_s positively influences the drawdown, indicating that a larger value of τ_s would have a larger drawdown.

3.3. Parameter Estimation for Field Test Data

The observed data obtained from an aquifer test conducted by the U.S. Geological Survey in Madison aquifer in the western Rapid City area, South Dakota (Greene, 1993) is analyzed herein based on the present solution. The Madison aquifer is consisted of limestone with the approximate thickness of 61 m and bounded by the Minnelusa Formation (a leaky confining layer) above and an impermeable layer beneath. The fractures and solution features in the aquifer are main factors increasing the permeability of this highly heterogeneous formation (Greene et al., 1999). A CRP test was conducted at the well RC-5 about 7 days. The pumping rate was maintained 386.11 m^3/h and the well almost fully penetrates the Madison aquifer with 11-inch-diameter (0.28 m) screen portion and 13.5-inch-diameter (0.34 m) casing portion. The drawdown data were measured at five wells. They are wells LC, SP-2, BHPL, CL-2, and CHLN-2 with distances 208 m, 518 m, 1,204 m, 2,713 m, and 3,566 m, respectively, from the well RC-5. Notice that all the distances are greater than twofold aquifer thickness; therefore, the partial penetration effect of observation wells are ignorable (Todd & Mays, 2005, p. 195).

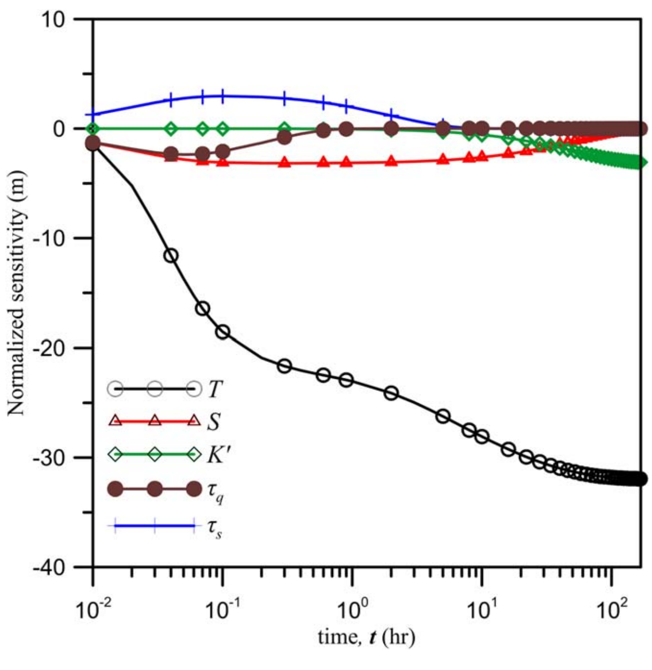

Figure 3. Temporal distribution curves of normalized sensitivity for five parameters observed at $r=10$ m.

Table 2 lists the aquifer parameters estimated by Greene (1993) via the type curve method developed by Hantush and Jacob's (1955) and the present solution with two cases for data obtained from four observation wells in South Dakota. Note that in the parameter estimation, the effects of wellbore storage and well skin are not considered and the present solution is coupled with the Levenberg-Marquardt algorithm given in the Mathematica routine FindFit (Wolfram, 1991). Case 1 only considers three parameters T, S, and K' while case 2 counts τ_q and τ_s in addition to those three parameters. The table lists the values of standard error of estimate (SEE) and mean error (ME) (Yeh, 1987). The SEE is defined as the square root of the sum of squared errors between the measured and predicted

Table 2
Parameters Estimated by Greene [1993] and Present Solution as Cases 1 and 2

Well	T (m^2/h)	$S \times 10^{-5}$	$K' \times 10^{-4}$ (m/h)	τq (h)	τs (h)	SEE (m)	ME (m)
Greene [1993]							
LC	6.19	10	0.26			0.577	0.217
SP-2	10.06	10	0.62			0.653	−0.368
BHPL	20.13	10	0.43			1.742	−1.620
CL-2	154.84	33	0.35			0.035	−0.027
CHLN-2	154.84	33	0.21			0.014	−0.0002
Case 1							
LC	5.32	11.04	1.61			0.299	0.067
SP-2	8.08	16.47	2.70			0.114	0.012
BHPL	18.64	51.20	5.38			0.041	−0.001
CL-2	116.13	41.70	1.94			0.011	−0.0005
CHLN-2	163.86	30.11	0.82			0.009	0.0002
Case 2							
LC	4.88	12.59	2.14	0.085	0.199	0.153	0.001
SP-2	7.20	17.55	3.25	0.203	0.383	0.103	−0.0005
BHPL	47.05	40.00	1.50	13.954	4.325	0.037	0.0003
CL-2	149.48	41.71	1.05	2.108	0.050	0.008	−0.0002
CHLN-2	186.27	28.82	5.40	8.362	7.148	0.009	0.0002

drawdowns divided by the degree of freedom, which is equal to the number of measured data minus the number of unknowns. The ME is defined as the mean of the sum of errors. Both statistics are used to assess the goodness of model fit to the observed data. On the basis of the values of SEE and ME, case 2 shows the present solution gives an excellent fit to the measured data. The SEE and ME values for case 2 are smaller than those for case 1 and Greene (1993), implying that the present solution can predict drawdown for pumping in the Madison aquifer. Both cases 1 and 2 give close estimates of the parameters T, S, and K' for data from LC and SP-2 which are located closer to the pumping well as shown in the bottom left corner of Figure 4. On the other hand, the estimated T is 116.13 m^2/h for CL-2 and 163.86 m^2/h for CHLN-2 in case 1 which are respectively much less than 149.48 m^2/h for CL-2 and 186.27 m^2/h for CHLN-2 in case 2. Note that both CL-2 and CHLN-2 are installed with distances 2,713 m and 3,566 m, respectively, from the pumping well. The estimated T for BHPL is 18.64 m^2/h in case 1 and 47.05 m^2/h in case 2 which are obviously different from those for LC and SP-2. This discrepancy may be attributed to the fact that the BHPL is located on the other side of the pumping well and its geology may be different from those of other observed wells in this highly heterogeneous formation. Additionally, it can be observed that τ_q is smaller than τ_s for LC and SP-2 indicating that the structural interaction is dominant. On the contrary, the estimation shows that τ_q is larger than τ_s for BHPL, CL-2, and CHLN-2 implying that the inertial force is significant. Figure 4 shows the measured data from those five wells and the temporal drawdown distributions predicted by Greene (1993) and the present solution with cases 1 and 2. Note that the drawdown curve for CL-2 is plotted in the bottom right plot to avoid overlapping with the curve of CHLN-2. The figure shows excellent fit for the present solution in case 2 to all the measured data from those five wells especially in the early time period. It is notable that Greene's (1993) predict drawdown curve at BHPL significantly deviates from the measured data as shown in the figure. The estimated leakage factor B ($= \sqrt{Tb'/K'}$) and r/B are respectively 1468.29 m and 0.82 by the present solution while the value of r/B is 0.4 in Greene (1993, p. 41). It can also be observed that the early-time drawdown data from BHPL obviously deviate from the type curve of $r/B = 0.4$ in Greene

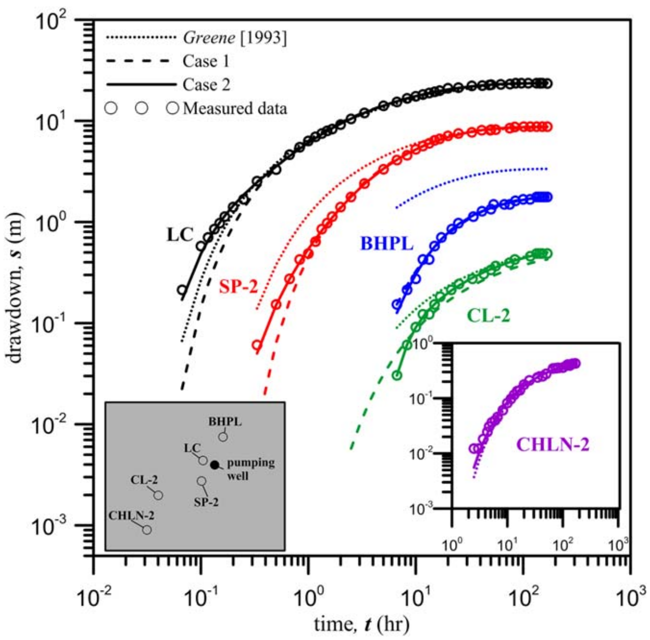

Figure 4. Measured data at five observation wells and temporal drawdown distributions predicted by Greene (1993) and present solution for cases 1 and 2.

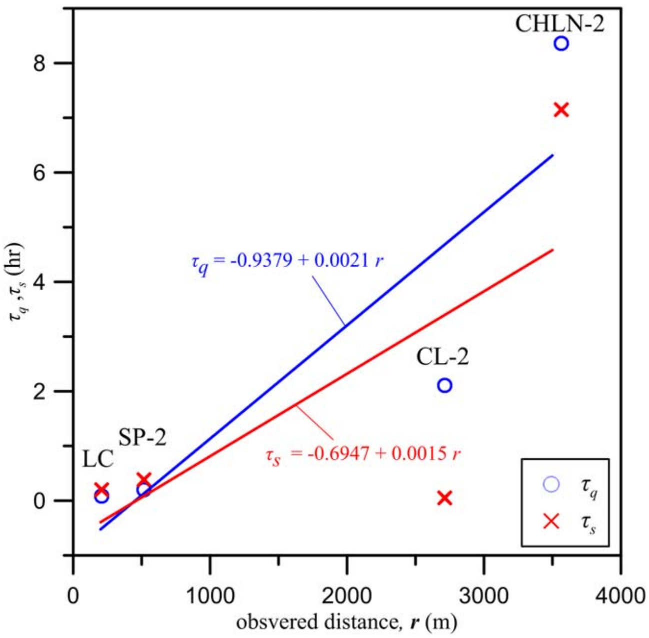

Figure 5. Linear regression lines for τ_q and τ_s against observed distances for wells excluding BHPL.

(1993), indicating that the graphical method would make large errors indicated in Figure 4 due to visual curve-fitting.

Figure 5 show the linear regression lines for τ_q and τ_s shown in Table 2 against observed distance for LC, SP-2, CL-2, and CHLN-2. Note that the data of τ_q and τ_s from BHPL are not included in the figure because it is on the opposite side of the other wells. This figure reveals that both phase lags τ_q and τ_s increase with the observed distance from the pumping well, indicating that the a larger distance from the pumping well may yield a longer lagging response for the flow. The nonlinear relationship indicated in the figure between the parameter value and observed distance may be due to the problems of the highly heterogeneous formation and/or non-straight arrangement of the observation wells.

4. Concluding Remarks

An analytical model for drawdown induced by CRP in a leaky confined aquifer with considering the phase lags in water flow and drawdown gradient is developed. The lagging behavior is introduced in flow equation to describe the drawdown induced by CRP. The model also incorporates the effects of wellbore storage and well skin. The drawdown solution is developed by the use of Laplace transform and Weber transform techniques. The sensitivity analysis is carried out to explore the influence of the change in each of the aquifer parameters on the drawdown. Additionally, the effects of dimensionless phase lags such as τ_{qD} for the water flow and τ_{sD} for the drawdown gradient on the drawdown are assessed. Furthermore, the present solution is applied to estimate the aquifer parameters via nonlinear least squares method for CRP test in a leaky confined fractured aquifer in South Dakota, USA. The results indicate that the present solution can not only be a useful tool to quantify the effects of inertia and structural interaction through the determination of the phase lag parameters τ_q and τ_s but also give a very good fit to the field observed data. The main findings are summarized as follows:

1. The mathematical expressions of the present lagging model and dual-porosity model are equivalent; however, the concepts in the model development are different. The former is constructed based on the lagging response due to the inertial and structural interaction effects. The latter, on the other hand, is developed with considering the flow exchange between the fracture and matrix in a fractured aquifer. However, the dual-porosity model should be subject to the constrain $S_mS_f/(\beta S_f+\beta S_m) < S_m/\beta$ (corresponding to $\tau_q < \tau_s$) because of the physical reality. Such a constrain implies that the dual-porosity model cannot deal with the problems for groundwater flow under the effect of the inertial force.
2. The parameters τ_{qD} and τ_{sD} in the present model are respectively introduced to reflect the lagging response induced by the inertial and structural interaction effects. When $\tau_{sD}=\tau_{qD}$, there is no lagging effect on the groundwater flow. As $\tau_{qD} < \tau_{sD}$, the structural interaction effect dominates the flow. A larger value of τ_{sD} would produce a larger drawdown before the dimensionless time close to τ_{sD} for a specific value of τ_{qD}. In contrast, a larger value of τ_{qD} would lead to a smaller drawdown before the time approaching τ_{qD} for given τ_{sD}. It is found that the drawdown curves are almost stabilized in the period for the dimensionless time from τ_{qD} to τ_{sD} . For the case that $\tau_{qD} > \tau_{sD}$, the inertial effect is dominant in the flow field. A larger value of τ_{sD} would cause larger drawdown before the time near τ_{sD} for a specific value of τ_{qD}. On the other hand, a larger value of τ_{qD} would result in smaller drawdown before the time approaching τ_{qD} with given τ_{sD}.
3. The results of sensitivity analysis show that the drawdown is very sensitive to the change in parameter T and relatively insensitive to the change in each of parameters of S, τ_q and τ_s. The effect of the phase lag on the drawdown is significant in the period associated with its lag time value and then gradually declines with time.
4. The present solution with estimated parameters gives very good fit to field observed drawdown data especially at the early operating time as compared with the model without considering the lagging

response. The results of parameter estimation for data from five different observation wells exhibit that there is an increasing trend in predicted phase lags with the observed distance from the pumping well.

Appendix A: The Derivation of Equation (24)

With neglecting the effect of wellbore storage (i.e., r_c or $C_D=0$), the inner boundary condition, equation (17), in Laplace domain can be written as

$$\frac{\partial \bar{s}_D(1,t_D)}{\partial r_D}=-\frac{2}{p} \tag{A1}$$

Applying the Weber transform to equation (16) results in

$$-a^2\tilde{\bar{s}}_D-\frac{4}{\pi ap}-\frac{\kappa\tilde{\bar{s}}_D}{(1+\tau_{sD}p)}=\frac{(1+\tau_{qD}p)}{(1+\tau_{sD}p)}p\tilde{\bar{s}}_D \tag{A2}$$

where $\tilde{\bar{s}}_D$ is the drawdown in Laplace-Weber domain and a is Weber variable. The inverse transform of Weber transform and associated formula can be found in Povstenko (2015, p. 22). The solution for drawdown can then be solved as

$$\tilde{\bar{s}}_D=\frac{4(1+\tau_{qD}p)}{\pi ap(a^2+p+\kappa+p^2\tau_{qD}+a^2\tau_{sD}p)} \tag{A3}$$

Employing the Laplace inverse transform to equation (A3), the drawdown solution in Weber domain can be obtained as

$$\tilde{s}_D=2\frac{\left\{-1+\beta_1-2\beta_1 e^{\frac{(1+\beta_2+a^2\tau_{sD})t_D}{2\tau_{qD}}}+a^2\tau_{sD}+2\kappa\tau_{sD}+[1+\beta_1-(a^2+2\kappa)\tau_{sD}]e^{\frac{\beta_1 t_D}{\tau_{qD}}}\right\}}{a\pi(a^2+\kappa)\beta_2}\times e^{-\frac{1+\beta_2+a^2\tau_{sD}t_D}{2\tau_{qD}}} \tag{A4}$$

Finally, the dimensionless transient drawdown solution without considering the wellbore storage can be obtained by applying Weber inverse transform (Povstenko, 2015, p. 22) as equation (24).

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