Physics of Processes

Periodic processes and periodicity

Periodic process

- an event that is regularly repeated after the same time - the so-called period

- it holds for any physical quantity X (t) describing a periodic process:

$$X(t) = X(t + nT)$$

- period $\boldsymbol{T}$, frequency $\boldsymbol{f = 1/T}$

- angular frequency $\boldsymbol{\omega} = 2\boldsymbol{\pi f}$

- periodic process are the basis for measuring time

- in the oldest period it was an observation of the periodicity of the movement of celestial bodies (Sun and Moon) across the sky - the origin of day, month and year

Periodicity of the solar system

The length of days and years on the planets of the solar system

Planet	The length of the day in multiples of the earth day	The length of the year in multiples of the earth year
Mercury	58,6	0,241
Venus	243	0,615
Mars	1,03	1,881
Jupiter	0,249	11,87
Saturn	0,425	29,45
Uranus	0,746	84,07
Neptune	0,795	164,9

Periodicity of the solar system

Example

Estimate the radius of the planetary orbit as a multiple of the Earth's orbit - Kepler's laws

Planeta	The length of the year in multiples of the earth year	Distance from the Sun in multiples of the Earth distance
Mercury	0,241	?
Venus	0,615	?
Mars	1,881	?
Jupiter	11,87	?
Saturn	29,45	?
Uranus	84,07	?
Neptune	164,9	?

$$F_G = F_D$$

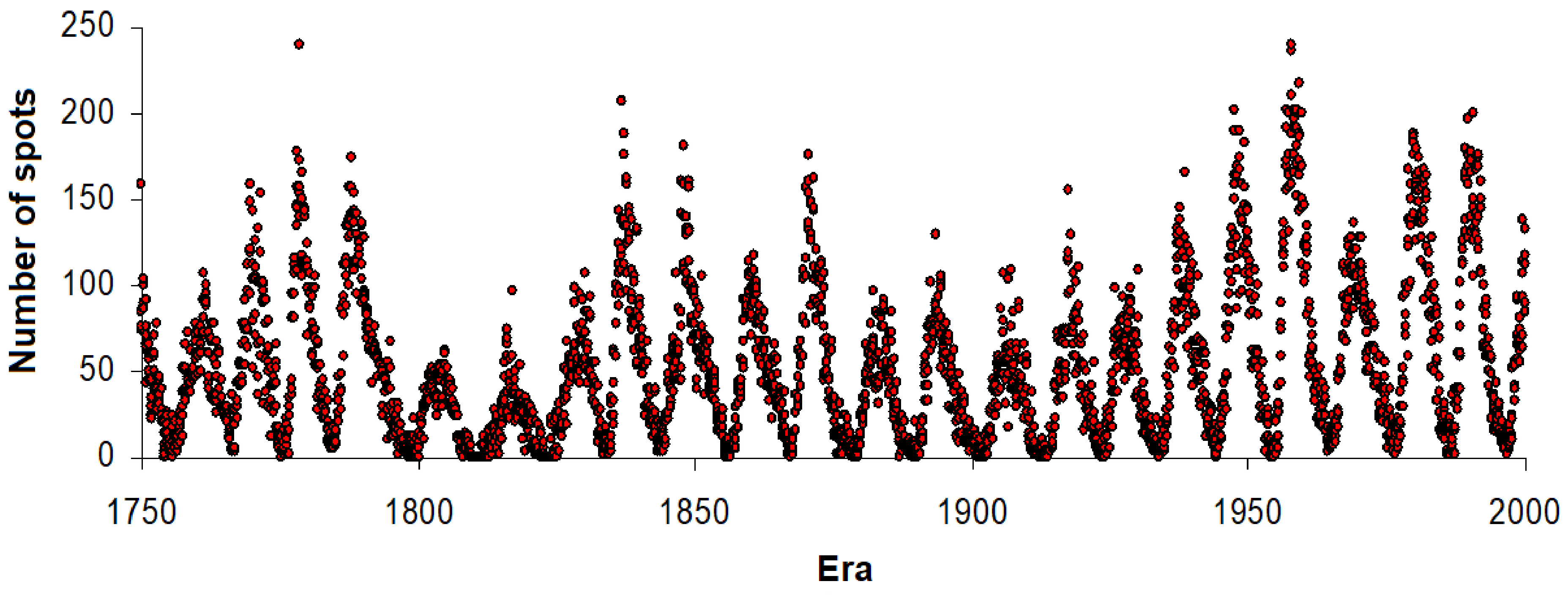
$$Tv = 2\pi r = T\sqrt{\frac{\kappa M_S}{r}}$$

$$KT = r^{\frac{3}{2}}$$

$$r / r_Z = (T / T_Z)^{\frac{2}{3}}$$

Periodicity of solar activity

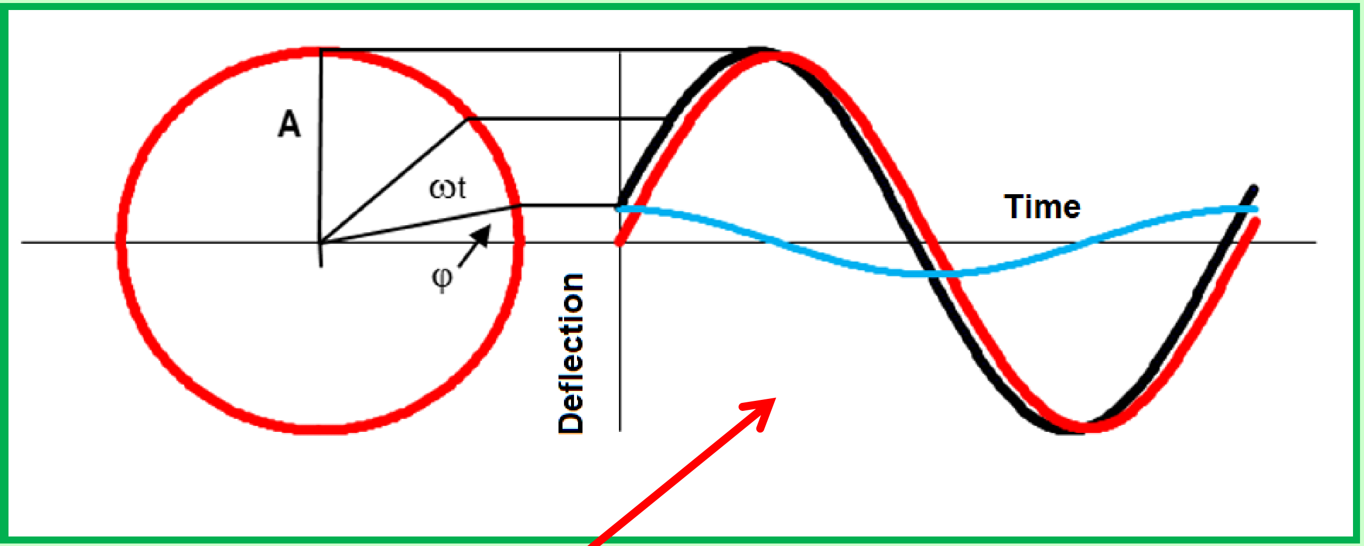
Sunspots - areas with lower surface temperature than the surroundings
- more complex periodic process
- described by an 11-year cycle



Harmonic process

- harmonic process is a periodic process described by a harmonic function, ie sine resp. cosine

- for describing the harmonic motion we can use the temporal evolution of a circular motion advantageously



***ω* – konst.**

x_0, y_0 coordinates of the starting point

$$y = A\sin(\omega t + \varphi) = x_0 \sin \omega t + y_0 \cos \omega t$$

$$x = A\cos(\omega t + \varphi) = x_0 \cos \omega t - y_0 \sin \omega t$$

$$A = \sqrt{x_0^{\ 2} + y_0^{\ 2}}$$

Notation with complex numbers

- an alternative description of the harmonic process

$$\hat{x}(t) = Ae^{i(\omega t+\varphi)} = A\cos(\omega t + \varphi) + iA\sin(\omega t + \varphi)$$

$$\mathrm{Re}\left(\hat{x}(t)\right) = A\cos(\omega t + \varphi)$$

$$\mathrm{Im}(\hat{x}(t)) = A\sin(\omega t + \varphi)$$

- we then express the *y*-component:

$$y(t) = \mathrm{Im}\,\hat{x}(t) = \mathrm{Im}\{Ae^{i\left(\omega t+\varphi\right)}\} = A\sin(\omega t + \varphi)$$

Complex numbers I

Definition

Complex number: $\hat{z} = x + iy$

Real and imaginary part: $x = \mathrm{Re}\,\hat{z}, \quad y = \mathrm{Im}\,\hat{z}$

Equality: $\hat{z}_1 = \hat{z}_2 \quad \Leftrightarrow \quad x_1 = x_2, y_1 = y_2$

Absolute value: $|\hat{z}| = \sqrt{x^2 + y^2} \geq 0$

Complex conjugate: $\bar{z} = x - iy$

Opposite complex number: $-\hat{z} = -x - iy$

Total: $\hat{z} = \hat{z}_1 + \hat{z}_2 = (x_1 + x_2) + i(y_1 + y_2)$

The difference is similar to the total

Addition laws: *commutative* $\hat{z}_1 + \hat{z}_2 = \hat{z}_2 + \hat{z}_1$

associative $\hat{z}_1 + (\hat{z}_2 + \hat{z}_3) = (\hat{z}_1 + \hat{z}_2) + \hat{z}_3$

Complex numbers II

Multiplication

Product of numbers: $\hat{z}_1 = x_1 + iy_1 \quad a \quad \hat{z}_2 = x_2 + iy_2$

Definition: $\hat{z} = \hat{z}_1\hat{z}_2 = \left(x_1x_2 - y_1y_2\right) + i\left(x_1y_2 + x_2y_1\right)$

Laws: *commutative* $\hat{z}_1\hat{z}_2 = \hat{z}_2\hat{z}_1$

associative $\hat{z}_1\left(\hat{z}_2\hat{z}_3\right) = \left(\hat{z}_1\hat{z}_2\right)\hat{z}_3$

distributive $(\hat{z}_1 + \hat{z}_2)\hat{z}_3 = \hat{z}_1\hat{z}_3 + \hat{z}_2\hat{z}_3$

If $\hat{z}_1 = \hat{z}_2 = i$, then it applies to the imaginary unit:

$$i^2 = i.i = \left(0 + 1i\right)\left(0 + 1i\right) = \left(0.0 - 1.1\right) + i\left(0.1 + 0.1\right) = -1$$

Complex numbers III

Division

Quotient of numbers $\hat{z}_1 = x_1 + iy_1 \quad a \quad \hat{z}_2 = x_2 + iy_2$

Definition: $$\hat{z} = \frac{\hat{z}_2}{\hat{z}_1} = \frac{\hat{z}_2 \bar{z}_1}{\hat{z}_1 \bar{z}_1} = \frac{\hat{z}_2 \bar{z}_1}{\left|\hat{z}_1\right|^2} = \frac{x_2 x_1 + y_2 y_1}{x_1^{\ 2} + y_1^{\ 2}} + i\,\frac{y_2 x_1 - x_2 y_1}{x_1^{\ 2} + y_1^{\ 2}} \qquad \hat{z}_1 \neq 0$$

Inverted number: $$\frac{1}{\hat{z}} = \frac{\bar{z}}{\left|\hat{z}\right|^2} = \frac{1}{x + iy} = \frac{1}{x + iy}\,\frac{x - iy}{(x - iy)} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

Consequences: $$\left|\hat{z}\right| = 0 \Leftrightarrow \hat{z} = 0$$

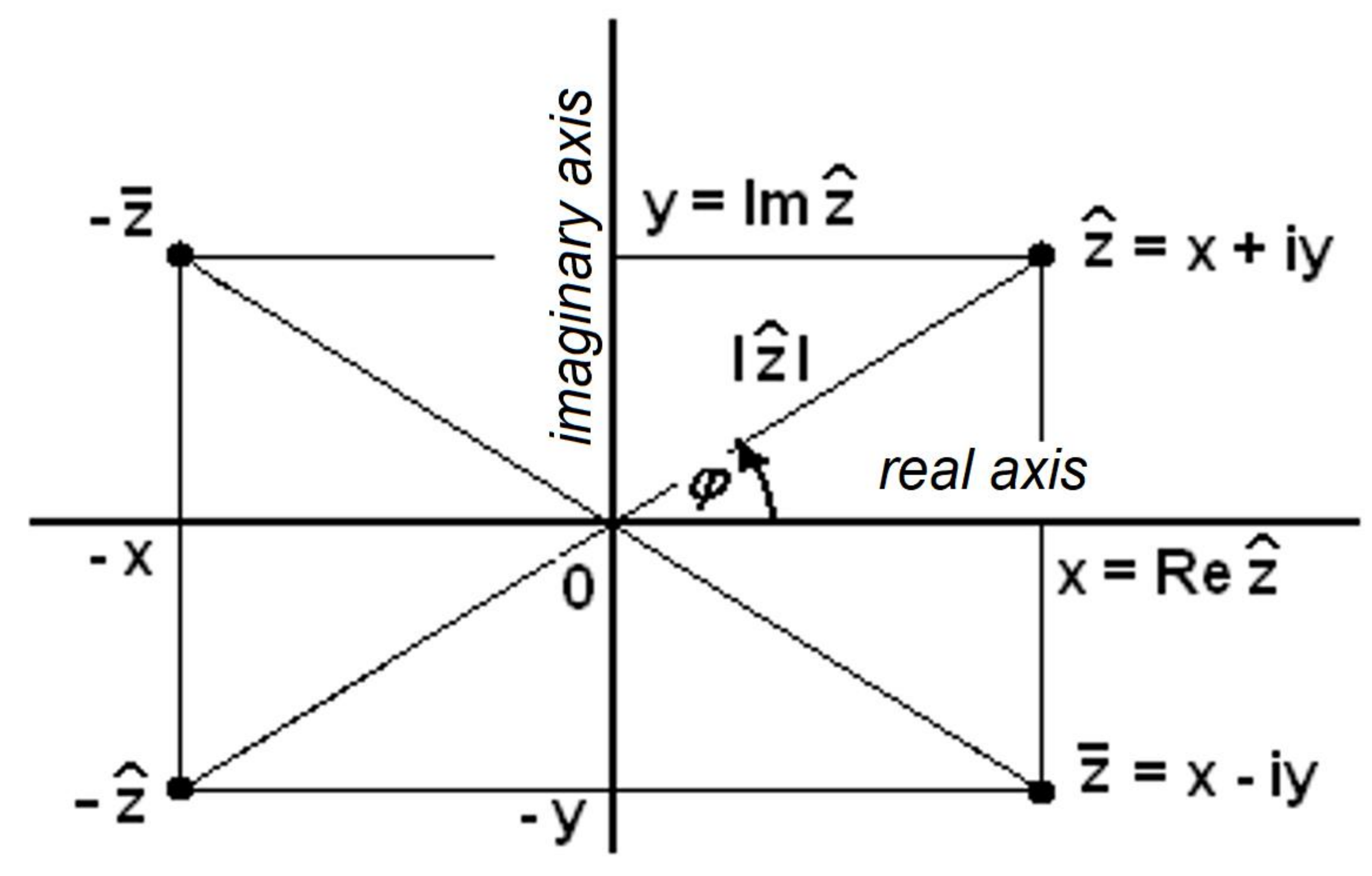
$$\left|\hat{z}_1 \hat{z}_2\right| = \left|\hat{z}_1\right|\left|\hat{z}_2\right|$$

$$\left|\hat{z}_1 \pm \hat{z}_2\right| \leq \left|\hat{z}_1\right| + \left|\hat{z}_2\right|$$

$$\left|\hat{z}_1 \pm \hat{z}_2\right| \geq \left|\hat{z}_1\right| - \left|\hat{z}_2\right|$$

Complex numbers IV

Gaussian plane, phase φ



Complex numbers V

Gaussian plane, phase φ, trigonometric and Euler notation

Phase (argument) of complex number $\varphi = \arg(\hat{z})$ is given:

$$\cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}$$

$\varphi = \varphi_0 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots.$

Trigonometric notation: $\hat{z} = x + iy = |\hat{z}|(\cos\varphi + i\sin\varphi)$

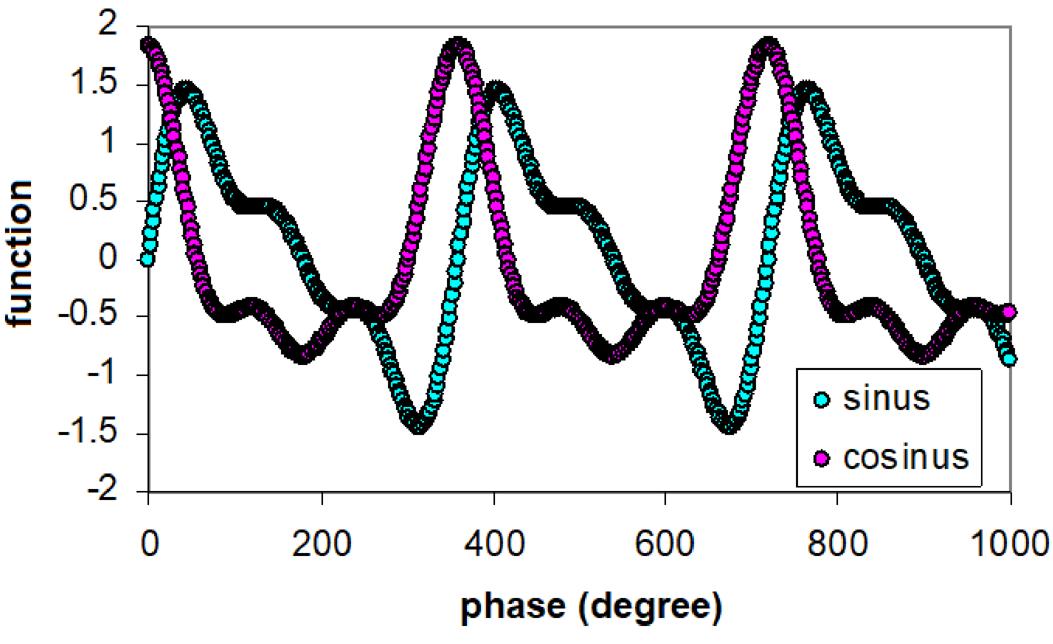
Euler notation: $\hat{z} = |\hat{z}| e^{i\varphi}$

Anharmonicity

- harmonic process - the simplest and most important type of periodic process

- the vast majority of real periodic process are not harmonic events - they are anharmonic process

- arbitrary, ie even anharmonic periodic process can be decomposed into a number of harmonic processes - this is enabled by harmonic (Fourier) analysis



$$A_s = \sin(\varphi) + \frac{1}{2}\sin(2\varphi) + \frac{1}{3}\sin(3\varphi)$$

$$A_c = \cos(\varphi) + \frac{1}{2}\cos(2\varphi) + \frac{1}{3}\cos(3\varphi)$$

Fourier (harmonic) analysis

Each periodic function *f* (*t*) with period *T* can be decomposed into a converging infinite series of harmonic functions:

$$f(t) = \sum_{k=0}^{\infty} \left[A_k \cos k\omega t + B_k \sin k\omega t \right], \quad \omega = \frac{2\pi}{T}$$

where the coefficients of the Fourier expansion are:

$$A_0 = \tfrac{1}{T} \int_0^T f(t) \mathrm{d}t \qquad B_0 = 0$$

for integer *k* = 1, 2, 3, ….

$$A_k = \frac{2}{T} \int_0^T f(t) \cos k\omega t \mathrm{d}t, \qquad B_k = \frac{2}{T} \int_0^T f(t) \sin k\omega t \mathrm{d}t$$

- Harmonic funktions in an infinite series have periods *T*, *T*/2, *T*/3… and angular frequencies ω, 2ω, 3ω …

- The development coefficients A_k and B_k usually decrease rapidly with increasing *k* - this allows to work with only a few components in the harmonic analysis

Fourier transform

- converts the function of time *s* (*t*) to a function of angular frequency $S(\omega)$:

$$F[s(t)] = S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} \mathrm{d}t = \left|S(\omega)\right| e^{i \arg S(\omega)}$$

- $S(\omega)$ is is the spectrum of the signal $s(t)$, which consists of the amplitude spectrum $\left|S(\omega)\right|$ and the phase spectrum $\arg S(\omega)$

- the classical Fourier transform is used for functions expressed in analytical form

- however, if we process specific measured values of a signal, we use discrete FT - DFT

Discrete Fourier transform
(DFT)

- it is the transformation of a signal from the time domain to the frequency domain

- ie the input to the DFT is a discrete sampled signal and the output is the discrete spectrum of this signal - information about the frequency components contained in it

- mathematically it is a transformation between sequences:

$d(k)$, k = 0, ..., N-1 a $D(n)$, n = 0, ..., N-1:

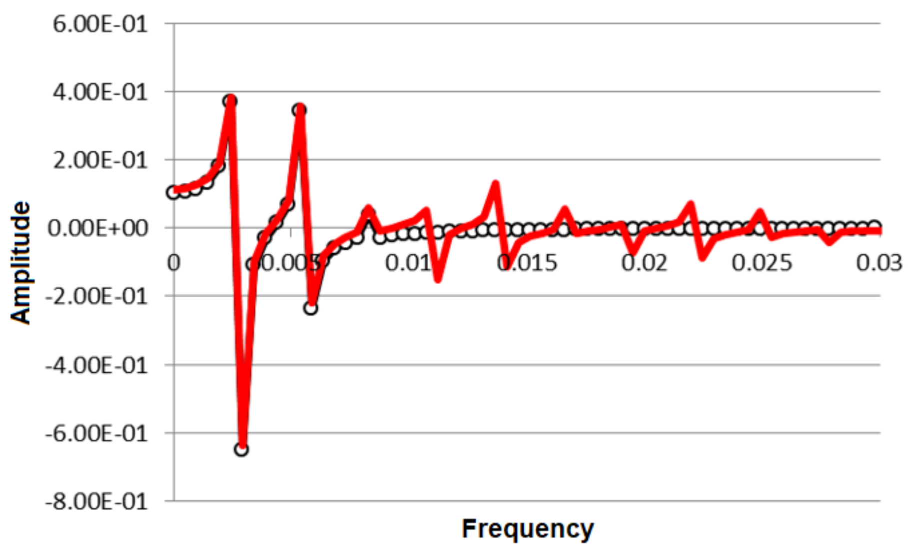
$$D(n) = \sum_{k=0}^{N-1} d(k) e^{-ink\,2\pi/N} \qquad n = 0, \ldots, N-1$$

$$d(k) = \frac{1}{N} \sum_{k=0}^{N-1} D(n) e^{ink\,2\pi/N} \qquad k = 0, \ldots, N-1$$

- today, the so-called Fast Fourier Transform (FFT) is used - a special DFT algorithm

FFT

FFT of functions reported for anharmonicity



Function sinus

Function cosinus