Physics of Processes

Free oscillating motion

- Oscillating motion - any repetitive action of the system in which the system does not exceed the final deviation from the relevant stable equilibrium position

- The cause of the oscillating motion is a reversible conservative force, which depends on the magnitude of the deflection of the system from the equilibrium position and is directed against this deflection (it acts so that the system returns to the equilibrium position)

- If the return force is directly proportional to the deviation from the equilibrium position, it is a linear return force

- The oscillations caused by this force are harmonic

- Linear harmonic oscillator - the cause of harmonic oscillations

- The oscillating system and the resulting forces can be very diverse in nature: elastic forces at the spring, gravitational force at the pendulum, induced electromotive force in the electric LC circuit, etc.

Linear harmonic oscillator

All harmonic oscillations are described by the equation:

$$\boxed{\begin{gathered}\ddot{u}(t)+\omega^2 u(t)=0\\ \frac{d^2u}{dt^2}+\omega^2 u=0\end{gathered}}$$

- ordinary differential equation of the 2nd order with constant coefficients and zero right side - homogeneous linear equation of the 2nd order

- *u*(*t*) - <u>immediate deviation of the system from equilibrium</u>

- a "colon" above this symbol - its second derivative according to time

- ω – **<u>inherent angular frequency of the oscillator</u>**

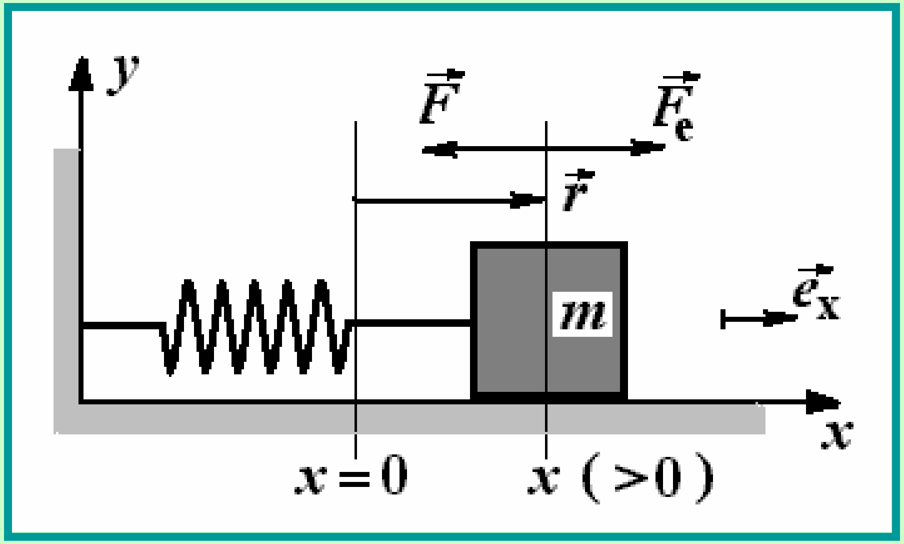
- The inherent angular frequency ω determines the "kinematics" of the deviations from the equilibrium system - practically this means that the process is periodic and *ω* is its angular frequency, so we define other <u>constant</u> quantities:

frequency $f = \omega/(2\pi)$

period $T = 1/f = 2\pi/\omega$

Mechanical harmonic oscillations (on a spring)

- one-dimensional (oscillation in one line)



- when the system is deviated from the equilibrium position, a linear return force begins to act

$$F(t) = -kx(t)$$

$$[k] = \frac{N}{m}$$

***k* -** force constant (spring stiffness)

$$m\ddot{\vec{r}}(t) = \vec{F}$$

$$m\ddot{x}(t) = -kx(t)$$

$$\ddot{x}(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

Special (one-dimensional) equation of motion for a linear harmonic oscillator

$$\boxed{\ddot{x}(t) + \omega^2 x(t) = 0}$$

$$\boxed{\omega^2 \equiv \frac{k}{m}, \quad \omega \equiv \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}, \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Dynamic image

Exact solution of the motion equation

$\boxed{\ddot{x}(t) + \omega^2 x(t) = 0}$ **- ordinary homogeneous linear differential equation of the second order with constant coefficients**

- assumed solution in the form $x(t) = e^{\lambda t}$

$\boxed{\lambda^2 + \omega^2 = 0}$ **- characteristic equation with solution $\lambda = \pm i\omega$**

$\boxed{x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}}$ **- general solution of differential motion equation**

where the integration constants C_1 and C_2 must be generally complex and $C_2 = \overline{C}_1$

$e^{\pm i\omega t} = \cos(\omega t) \pm i.\sin(\omega t)$, $A_1 = C_1 + C_2 = A \sin \varphi$, $A_2 = i(C_1 - C_2) = A \cos \varphi$

$\boxed{x(t) = A_1 \cos \omega t + A_2 \sin \omega t}$ **General equation of linear harmonic motion**

The constants A_1 and A_2 are understood as components of the resulting motion, where the component represented by the constant A_1 represents a motion in which the deviation takes the value A_1 for time equal to zero and the component represented by the constant A_2 represents the motion shifted by π/2 relative to the previous motion; therefore applies:

Alternative expression for the solution

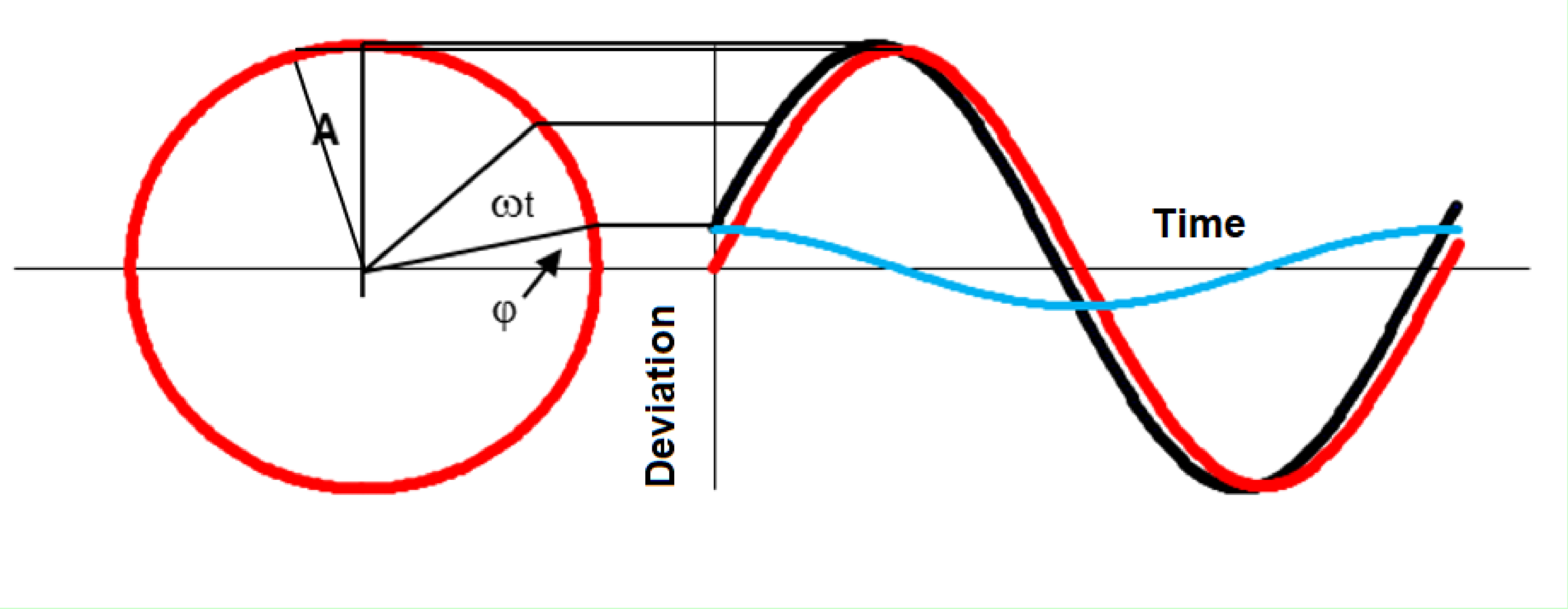
$$\boxed{A^2 = A_1^{\,2} + A_2^{\,2}} \longrightarrow \boxed{x(t) = A\sin(\omega t + \varphi)}$$

Periodicity:

$\boxed{x(t + T) = x(t)}$

$$A\sin[\omega(t+T)+\varphi] = A\sin(\omega(t)+\varphi)$$
$$\omega(t+T)+\varphi = \omega t + \varphi + 2\pi$$
$$\omega T = 2\pi$$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$



Speed and acceleration

$$x(t) = A\sin(\omega t + \varphi)$$

Derivation of relations for velocity and acceleration by successive derivation

$$v(t) = \dot{x}(t) = A\omega\cos(\omega t + \varphi) = v_m\cos(\omega t + \varphi)$$

$$a(t) = \ddot{x}(t) = -A\omega^2\sin(\omega t + \varphi) = -\omega^2 x(t) = a_m\sin(\omega t + \varphi)$$

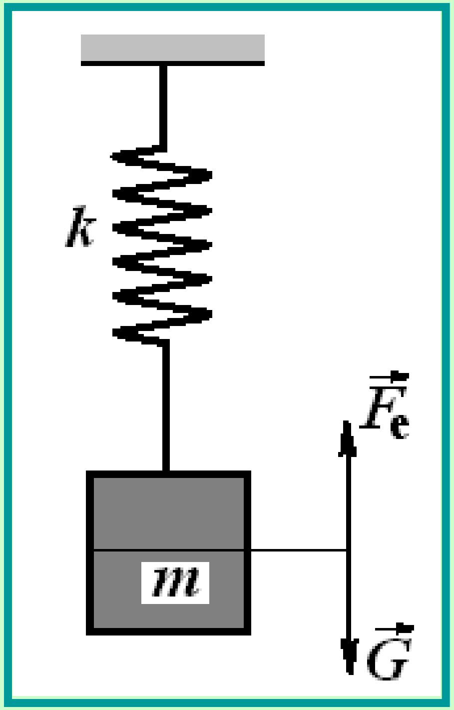
The velocity is shifted by π / 2 with respect to the deflection, the acceleration is shifted by π with respect to the deflection

$$v_m = A\omega$$

$$a_m = -A\omega^2$$

- **if the oscillating system is not affected by dissipative forces, the oscillations are undamped**

Linear oscillator in the gravitational field



$$\boxed{F = -ku + mg = m\ddot{u}}$$

Equation of motion

$$\ddot{\bar{u}}(t) + \omega^2 \bar{u}(t) = 0 \qquad \bar{u}(t) = u(t) - g\omega^{-2}$$

$$u(t) = \bar{u}(t) + g\omega^{-2} = g\omega^{-2} + A\sin(\omega t + \varphi)$$

Linear harmonic oscillator in the gravitational field:

Oscillations with frequency ω around the equilibrium position u_R:

$$\boxed{u_R = g\omega^{-2} = \frac{mg}{k}} \qquad \boxed{\omega = \sqrt{k/m}}$$

Energy of undamped harmonic oscillations

- in the equilibrium position we assume potential energy $E_p = 0$

- the potential energy E_p corresponding to the deviation $\vec{u}(t)$ is given by the work performed by external force $\vec{F}^*$, which overcomes the internal elastic force of the systém $\vec{F} = -k\vec{u}$ to bring the system out of equilibrium

- is therefore $\vec{F}^* = -\vec{F}$

Energy of undamped harmonic oscillations

Potential energy:

$$dA = \vec{F}^* d\vec{u} = -\vec{F} d\vec{u} = -F du$$

$$E_p = \int_0^u dA = -\int_0^u F du' = \int_0^u k u' du' = \frac{1}{2} k u^2$$

$$u(t) = U \sin(\omega t + \varphi)$$

$$\boxed{E_p = \frac{1}{2} k U^2 \sin^2(\omega t + \varphi) = \frac{1}{2} m \omega^2 U^2 \sin^2(\omega t + \varphi)}$$

Kinetic energy:

$$v(t) = \dot{u}(t) = U \omega \cos(\omega t + \varphi)$$

$$\boxed{E_k = \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 U^2 \cos^2(\omega t + \varphi)}$$

The total energy is constant, independent of time:

$$\boxed{E = E_p + E_k = \frac{1}{2} m \omega^2 U^2 \left[\sin^2(\omega t + \varphi) + \cos^2(\omega t + \varphi)\right] = \frac{1}{2} m \omega^2 U^2}$$

- the law of conservation of mechanical energy applies to undamped oscillations

Potential of a linear harmonic oscillator

$$\boxed{\bar{u} = \frac{E_p}{m} = \frac{1}{2}\omega^2 U^2 \sin^2(\omega t + \varphi) = \frac{1}{2}(u\omega)^2 = -\frac{1}{2}au}$$