Physics of Processes

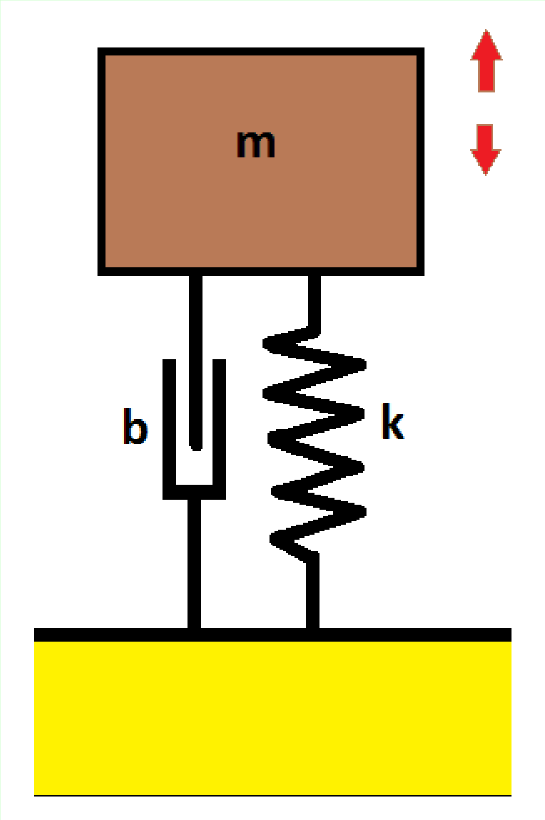
Damped and forced harmonic oscillator

Damping of oscillating systems

- in real systems with dissipative forces, oscillating motion is limited or directly eliminated
- after one oscillation is performed, the system does not return to its original state
- therefore, it is not a strictly periodic process
- only in systems with a low level of damping we can consider the process to be quasi-periodic with damping as a "fault"
- tlumení damping is a common accompanying phenomenon of any real oscillating motion or process, which is amplified or suppressed as needed
- a frequent example of vibration damping in technical systems and machines in general
- simulation

Linear harmonic oscillator

Damped



- arises from a linear harmonic oscillator by parallel assignment of a damper with a coefficient of linear resistance *b*

- the simplest case:
the braking resistance force is directly proportional to the oscillation speed

Damping force proportional to speed

$$\vec{F}_b = -b\vec{v}, \quad b \succ 0$$

F_b braking (resistive) force,

b coefficient of linear resistance

$$m\ddot{x} = F + F_b$$

$$m\ddot{x} = -kx - b\dot{x}$$

equation of motion

δ decrement of attenuation

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{k}{m}}, \quad \delta = \frac{b}{2m}$$

The equation of motion is again an ordinary linear differential equation of the 2nd order with constant parameters and zero right side.

$$x(t) = e^{\lambda t}$$

solution shape

$$\lambda^2 + 2\delta\lambda + \omega^2 = 0$$

characteristic equation

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2} = -\delta \pm D \qquad D^2 = \delta^2 - \omega^2$$

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

general solution: equation (1)

C_1, C_2 – from the initial conditions of motion

λ_1, λ_2 – generally complex numbers

4 solution cases:

I. Negligible attenuation

$\delta/\omega \ll 1$

$D^2 \sim -\omega^2$

- this state can be characterized as a case of almost undamped harmonic oscillations

$$\tilde{\omega}^2 = \omega^2 = k/m$$

- resulting angular frequency of damped oscillations = inherent angular frequency of the oscillator

- the change in the frequency of the oscillator is in this case caused by changes in some of the quantities *k* and *m*

II. Damped periodic oscillations

(0 < *δ*/*ω* < 1)

In this case, D^2 is negative and the roots of the characteristic equation are complex numbers:

$$\lambda_{1,2} = -\delta \pm i\sqrt{\omega^2 - \delta^2} = -\delta \pm i\tilde{\omega}, \quad \tilde{\omega} = \sqrt{\omega^2 - \delta^2}$$

$\tilde{\omega}^2 = \frac{k}{m} - \delta^2$ **is the angular frequency of the damped harmonic oscillations, which is less than the angular frequency *ω***

$$x(t) = C_1 e^{(-\delta + i\tilde{\omega})t} + C_2 e^{(-\delta - i\tilde{\omega})t}$$ **- solution of the equation (1)**

$$x(t) = e^{-\delta t}\left[C_1 e^{i\tilde{\omega}t} + C_2 e^{-i\tilde{\omega}t}\right] = Ae^{-\delta t}\sin(\tilde{\omega}t + \varphi)$$

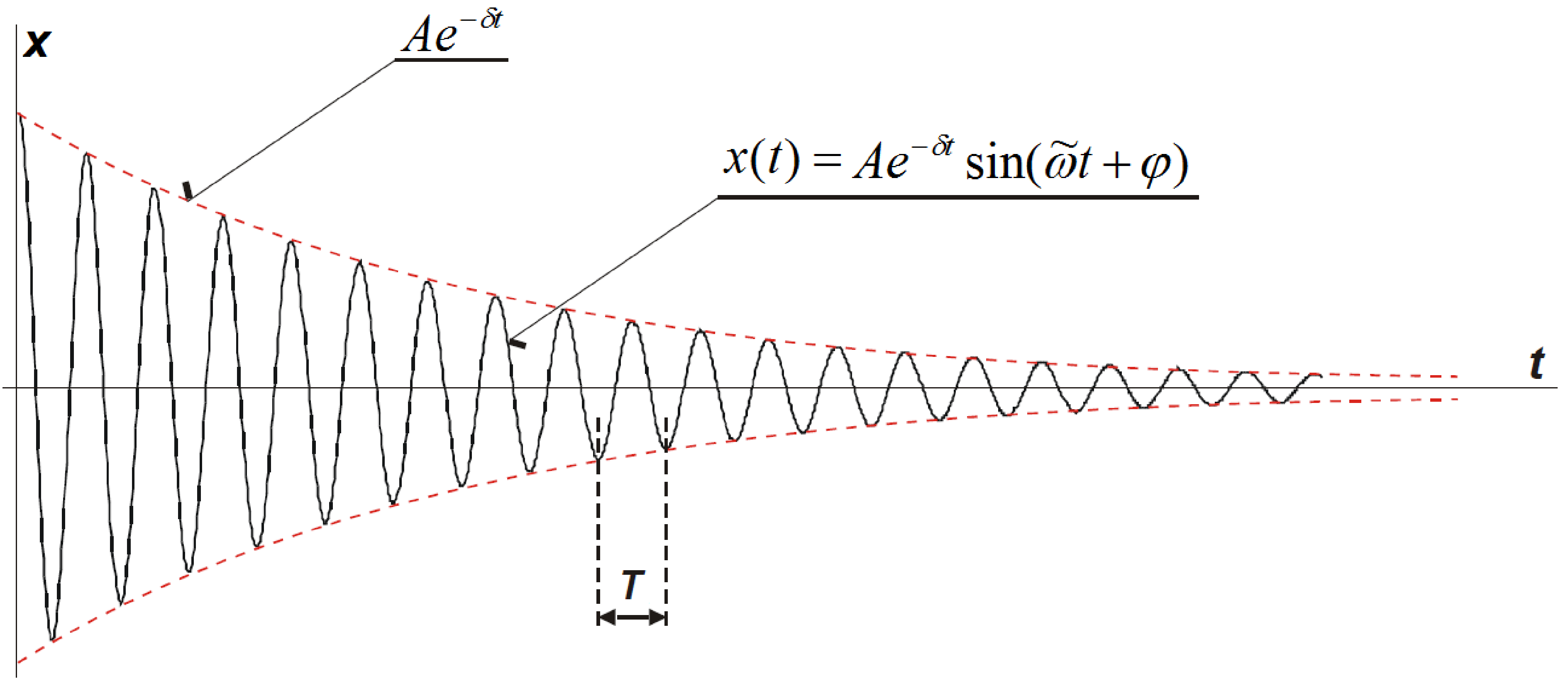
A, φ **- new integration constants - determined from initial conditions**

<u>Damped periodic oscillations</u>

- it is not strictly a periodic process - the amplitude of oscillations decreases exponentially over time

II. Damped periodic oscillations

($0 < \delta/\omega < 1$)



Velocity and acceleration

$$x(t) = Ae^{-\delta t}\sin(\tilde{\omega}t + \varphi)$$

$$v(t) = \frac{dx}{dt} = A\tilde{\omega}e^{-\delta t}\cos(\tilde{\omega}t + \varphi) - A\delta e^{-\delta t}\sin(\tilde{\omega}t + \varphi) =$$
$$= Ae^{-\delta t}\left[\tilde{\omega}\cos(\tilde{\omega}t + \varphi) - \delta\sin(\tilde{\omega}t + \varphi)\right]$$

simulation

$$a(t) = \frac{dv}{dt} = -A\delta e^{-\delta t}\left[\tilde{\omega}\cos(\tilde{\omega}t + \varphi) - \delta\sin(\tilde{\omega}t + \varphi)\right] +$$
$$+ Ae^{-\delta t}\left[-\tilde{\omega}^2\sin(\tilde{\omega}t + \varphi) - \delta\tilde{\omega}\cos(\tilde{\omega}t + \varphi)\right] =$$
$$= Ae^{-\delta t}\left[-\delta\tilde{\omega}\cos(\tilde{\omega}t + \varphi) + \delta^2\sin(\tilde{\omega}t + \varphi) - \tilde{\omega}^2\sin(\tilde{\omega}t + \varphi) - \delta\tilde{\omega}\cos(\tilde{\omega}t + \varphi)\right] =$$
$$= Ae^{-\delta t}\sin(\tilde{\omega}t + \varphi)\left[\delta^2 - \tilde{\omega}^2\right] - 2Ae^{-\delta t}\delta\tilde{\omega}\cos(\tilde{\omega}t + \varphi) =$$
$$= x(t)[\delta^2 - \tilde{\omega}^2] - 2Ae^{-\delta t}\delta\tilde{\omega}\cos(\tilde{\omega}t + \varphi)$$

Attenuation

- the ratio of 2 consecutive maximum deviations to the same side from the equilibrium position:

$$\frac{x_2}{x_1} = \frac{Ae^{-\delta(t_1+T)}\sin\left[\tilde{\omega}(t_1+T)+\varphi\right]}{Ae^{-\delta t_1}\sin\left[\tilde{\omega}t_1+\varphi\right]} = e^{-\delta T} = const.$$

– attenuation

$$\vartheta = \ln\left(x_i / x_{i+1}\right) = \delta T = 2\pi\frac{\delta}{\tilde{\omega}}$$

logarithmic attenuation decrement

- when comparing damped and undamped oscillations, applies:

$$\frac{\omega_0}{\tilde{\omega}} = \frac{T}{T_0} = \sqrt{1+\frac{\vartheta^2}{4\pi^2}}$$

Influence of attenuation on the change of periodicity of damped oscillations:

Attenuation	ϑ	Period ratio
1	0	1
0,5	0,6931	1,0061
0,1	2,3026	1,0650
0,05	2,9957	1,1078
0,01	4,6052	1,2398

$$\tau = 1/\delta$$

- damping time constant

- expresses the time during which the amplitude of the oscillations decreases e-times:

$$x(t) = e^{-t/\tau}\left[C_1 e^{i\tilde{\omega}t} + C_2 e^{-i\tilde{\omega}t}\right] = Ae^{-t/\tau}\sin(\tilde{\omega}t+\varphi)$$

III. Critical attenuation

- astatic motion ($\delta/\omega = 1$)

If $\delta/\omega = 1$, then $D = 0$ and the solution of the characteristic equation is a double root $\lambda_{1,2} = -\delta$. The motion ceases to be a periodic motion.

The general solution of the equation of motion has the form:

$$x(t) = e^{-\delta t}(C_1 + C_2 t)$$

If the oscillator at the moment $t = 0$ has a deviation from the equilibrium position $x_0 = A$ and zero velocity $v_0 = 0$, the equation gives:

$$x(t) = e^{-\delta t}(C_1 + C_2 t), \quad x(0) = C_1 = A$$

$$v(t) = \frac{dx}{dt} = C_2 e^{-\delta t} - \delta(C_1 + C_2 t)e^{-\delta t}, v(0) = C_2 - \delta C_1 = 0$$

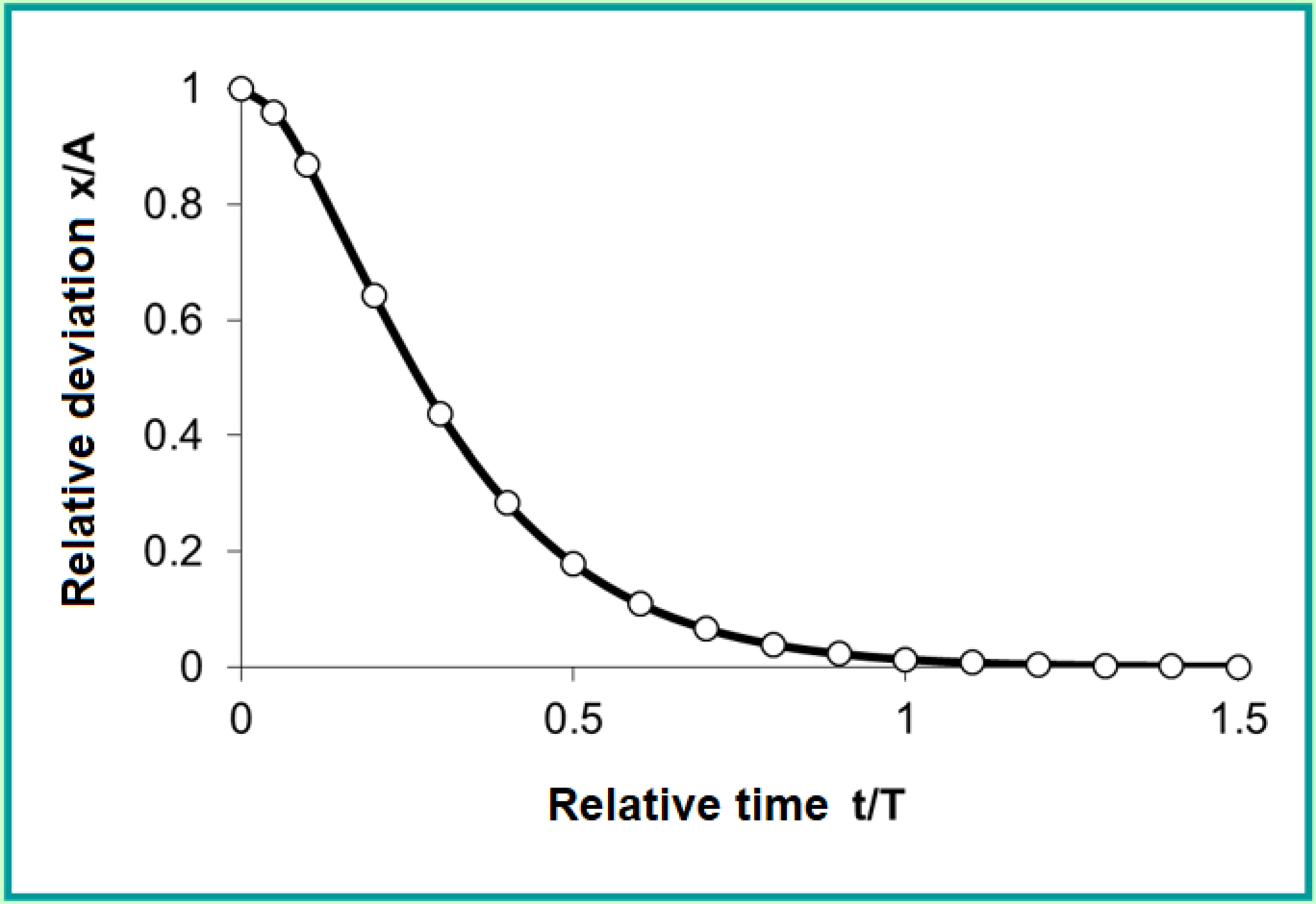
$$C_2 = \delta C_1 = \delta A$$

So it applies:

$$x(t) = e^{-\delta t}(C_1 + C_2 t) = Ae^{-\delta t}(1 + \delta t)$$

Critical attenuation is used to dampen motion wherever it is necessary to establish an equilibrium position as quickly as possible.

III. Astatic motion



IV. Overdamped oscillations - strong attenuation

($\delta/\omega > 1$)

If $D^2 = \delta^2 - \omega^2 > 0$, then both roots of the characteristic equation are real and positive. The general solution of equation (1) then has a nonharmonic character. In this case, the damping is so great that the system returns to the equilibrium position only very slowly after deviating. The so-called *aperiodic (overdamped) motion* takes place.

$$x(t) = C_1 e^{(-\delta+D)t} + C_2 e^{(-\delta-D)t} = e^{-\delta t}\left(C_1 e^{Dt} + C_2 e^{-Dt}\right)$$

$$x(t) = Ae^{-\delta t}\sinh(Dt+\varphi) = Ae^{-\delta t}\frac{e^{(Dt+\varphi)} - e^{-(Dt+\varphi)}}{2} =$$

$$= Ae^{-\delta t}\frac{e^{(Dt+\varphi)} - e^{-(Dt+\varphi)}}{2} = Ae^{-\delta t}\frac{e^{\left(\sqrt{\left(\delta^2-\omega^2\right)}t+\varphi\right)} - e^{-\left(\sqrt{\left(\delta^2-\omega^2\right)}t+\varphi\right)}}{2}$$

- quantities C_1, C_2 and A, φ are determined by the initial conditions as in other cases

Linear harmonic oscillator

with driving force and damping

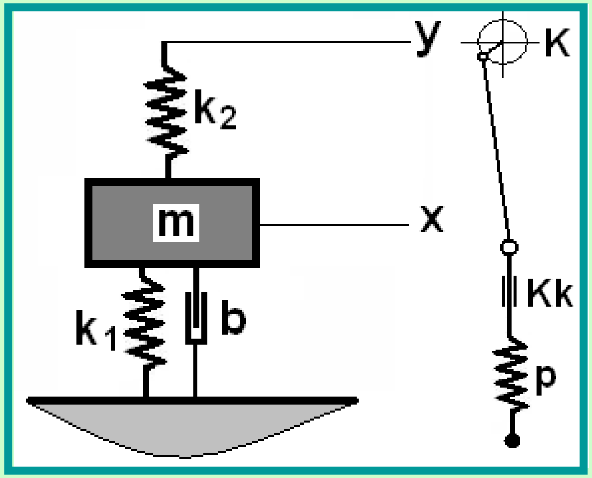
<u>Driving force:</u> Forced oscillating motion is performed by the system due to the action of a time-varying, usually periodic, external driving force:

$$\vec{F}_B(t) = \vec{B}\sin(\Omega t + \beta)$$

- inherent frequency of system *ω*, frequency driven harmonic oscillations *Ω*

$$m\ddot{x} = -kx - b\dot{x} + B\sin(\Omega t + \beta)$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \frac{B}{m}\sin(\Omega t + \beta)$$ **-** equation of motion



- ordinary linear differential equation of the 2nd order with constant coefficients, inhomogeneous

- its solution can be written as the sum of the general solution x_b(*t*appropriate homogeneous equations and any particular solution $x_p(t)$ of the whole equation

$$x_p(t) = C\sin(\Omega t + \beta + \gamma)$$

x(t) = x_b(t) + x_p(t)

$$x(t) = x_b(t) + x_p(t) = Ae^{-\delta t}\sin\left[\left(\omega^2 - \delta^2\right)t + \varphi\right] + C\sin(\Omega t + \beta + \gamma)$$

Solution

Simulation

The deviation of the damped motion decreases exponentially with increasing time and <u>after a certain time, practically only undamped oscillations remain,</u> whose frequency is equal to the frequency of the driving oscillations *Ω* of external driving force. Then the oscillator performs only <u>oscillations with the same frequency</u> as the frequency of the external periodic variable force acting on it and these oscillations are called forced oscillations.

The agreement of the solutions with the differential equation for the argument of the function sin:

0 $$2m\delta C\Omega = -B\sin\gamma$$

π/2 $$-mC\Omega^2 + mC\omega^2 = B\cos\gamma$$

$$\operatorname{tg}\gamma = -\frac{2\delta\Omega}{\omega^2 - \Omega^2}$$

$$C = \frac{B}{m\sqrt{\left(\omega^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

$$\sin\gamma = -\frac{2\delta m\Omega C}{B}$$

Consequences of the solution

- **after a certain time from the beginning of the force, a steady state occurs and the system already oscillates periodically**
- **the inherent oscillations of the system subside after a certain time and the system stabilizes at an undamped harmonic oscillation with frequency *Ω***
- **however, the effect of damping is reflected both in the amplitude of the driven oscillations – see:**

$$C = \frac{B}{m\sqrt{\left(\omega^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

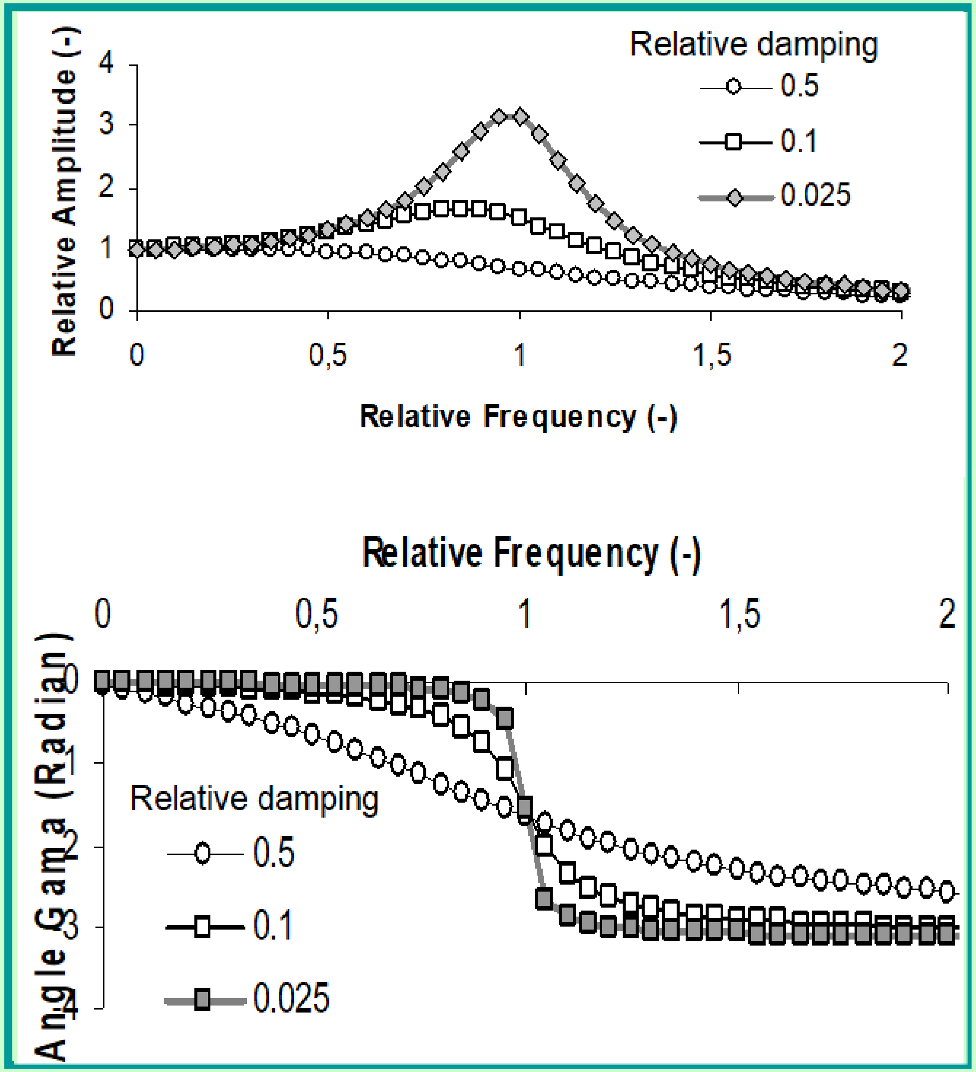
and in their phase – see: $\mathrm{tg}\,\gamma = -\frac{2\delta\Omega}{\omega^2 - \Omega^2}$ **and** $\sin\gamma = -\frac{2\delta m\Omega C}{B}$

- **both constants *A, φ*, present in the solution of the equation**

$$x(t) = x_b(t) + x_p(t) = Ae^{-\delta t}\sin\left[\left(\omega^2 - \delta^2\right)t + \varphi\right] + C\sin\left(\Omega t + \beta + \gamma\right)$$

lose their effect during damping as $x_b(t) \to 0$

Resonance of driven oscillations



The difference $\omega^2 - \Omega^2$ in the denominator of the expression for C shows that the amplitude C of the forced oscillations is the larger by what the smaller the difference between the angular frequency ω of its own undamped oscillations and the angular frequency Ω of the forced oscillations. It is *resonance.* The dependence of the amplitude of the forced oscillations on the frequency of the driving oscillations $C = C(\Omega)$ is called the *resonance curve in amplitude*.

Quality factor and energy of driven oscillations

Q – characteristics of the oscillating system

$$Q = 2\pi \frac{\overline{W}}{\Delta_T W}$$

2π times the ratio of the average energy of the oscillator $\overline{W}$ to the energy $\Delta_T W$ dissipated by the damping force over a period of one period

$$\overline{W} = \frac{1}{2} m v_m{}^2 = \frac{1}{2} m\Omega^2 C^2 = \frac{1}{2}\frac{\Omega^2}{m}\frac{B^2}{\left(\omega^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}$$

$$\Delta_T W = -\int_0^T F_t dx = 2\delta m \Omega^2 C^2 \int_0^T \cos^2 \Omega t dt$$

$$\Delta_T W = 2\pi\delta m \Omega C^2 = 2\pi \frac{B^2}{m}\frac{\delta\Omega}{\left(\omega^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}$$

$$Q = 2\pi \frac{\overline{W}}{\Delta_T W} = \frac{\Omega}{2\delta}$$

- **at a large value of the quality factor Q, the <u>accumulated energy is significantly greater</u> than the energy supplied by the oscillation during one cycle**
- **each oscillator is fully characterized by its own frequency and quality factor, regardless of the oscillating process and the nature of the power dissipation**

Resonance at low damping

($\delta \ll \omega$)

- at low damping values δ can be neglected and we consider the resonance to be the case when $\omega = \Omega$

- the phase shift γ between the driving force and the forced oscillations in the resonant state is just $-\pi/2$ - see the graph

- this phase shift ensures that the driving force always acts in the direction of "moving" the oscillator and does a positive work

Stabilization:

- all the work done by the driving force is first used to overcome the resistances that dampen the oscillations - the amplitude of the oscillations and their energy increases

- then the amplitude and approximately also the energy reach a stable maximum value