Physics of Processes

Superposition of oscillations and wave formation

Superposition of oscillations

The free, damped and forced oscillatory motion is described by linear differential equations. The consequence is the principle of superposition. The principle of superposition means that the individual oscillations that take place simultaneously in a (linear) system happen independently of each other. The resulting composite oscillations of the system are then the superposition of all its oscillations occurring simultaneously. This is true for both scalars and vectors.

Superposition of oscillations with the same direction and the same frequency

$$u(t) = u_1(t) + u_2(t) \qquad u_1(t) = A_1 \sin(\omega t + \varphi_1), \quad u_2(t) = A_2 \sin(\omega t + \varphi_2)$$

- the resulting deflection *u* (*t*) from the equilibrium position is again a harmonic function

$$\boxed{u(t) = A\sin(\omega t + \varphi) = A\sin\omega t\cos\varphi + A\cos\omega t\sin\varphi}$$

$$u_1(t) + u_2(t) = A_1 \sin\omega t\cos\varphi_1 + A_1\cos\omega t\sin\varphi_1 + A_2\sin\omega t\cos\varphi_2 + A_2\cos\omega t\sin\varphi_2$$

$$\boxed{\begin{aligned} A\cos\varphi &= A_1\cos\varphi_1 + A_2\cos\varphi_2 \\ A\sin\varphi &= A_1\sin\varphi_1 + A_2\sin\varphi_2 \end{aligned}} \rightarrow \boxed{\begin{aligned} A^2 &= A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1) \\ tg\,\varphi &= \frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2} \end{aligned}}$$

Superposition of oscillations

- **the equation** $A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)$ **implies:**
- **if the oscillations are in phase, i.e. $\varphi_2 - \varphi_1 = 0$, then $A = A_1 + A_2$ – amplitudes add up, oscillations amplify**
- **if the oscillations are in the opposite phase, i.e. $\varphi_2 - \varphi_1 = \pi$, then $A = A_1 - A_2$ – amplitudes are subtracted, oscillations are attenuated**
- **if $A_1 = A_2$, then $A = 0$ – both oscillations will suppress each other**

Superposition of oscillations

<u>Superposition of oscillations with the same direction and different frequencies</u>

$$u_1(t) = A_1 \sin(\omega_1 t + \varphi_1), \qquad u_2(t) = A_2 \sin(\omega_2 t + \varphi_2)$$

- we set the beginning of the oscillatory motion tracking to the moment t_0, when both partial oscillations have the same phase: $\omega_1 t_0 + \varphi_1 = \omega_2 t_0 + \varphi_2 = \varphi$. Then:

$$u(t) = u_1(t) + u_2(t) = A_1 \sin(\omega_1 t + \varphi) + A_2 \sin(\omega_2 t + \varphi)$$

The resulting oscillation is periodic only if the ratio of the frequencies f1 and f2 of both partial oscillations is given by the ratio of integers n_1 and n_2:

$$\boxed{\frac{\omega_1}{\omega_2} = \frac{f_1}{f_2} = \frac{T_2}{T_1} = \frac{n_1}{n_2}}$$

When this condition is met, the resulting oscillation will be described by a periodic and generally inharmonic function that repeats after the shortest period of time that includes n_1 oscillations with period T_1 and n_2 oscillations with period T_2. This period T is the smallest common multiple of both periods T_1 and T_2. If the T_2/T_1 ratio is irrational, the two superposed oscillations cannot meet so that the motion cannot repeat and the motion cannot be periodic.

Superposition of oscillations

Beats

- **superposition of oscillations with the same direction and different frequencies, which do not differ much from each other, i.e. they apply** $\omega_1 \neq \omega_2$, $|\omega_1 - \omega_2| << \omega_1 + \omega_2$
- **the condition must be met** $\boxed{\dfrac{\omega_1}{\omega_2} = \dfrac{f_1}{f_2} = \dfrac{T_2}{T_1} = \dfrac{n_1}{n_2}}$

- **we will further assume that** $\omega_1 > \omega_2$**, and the same amplitudes** $A_1 = A_2 = A$**.**

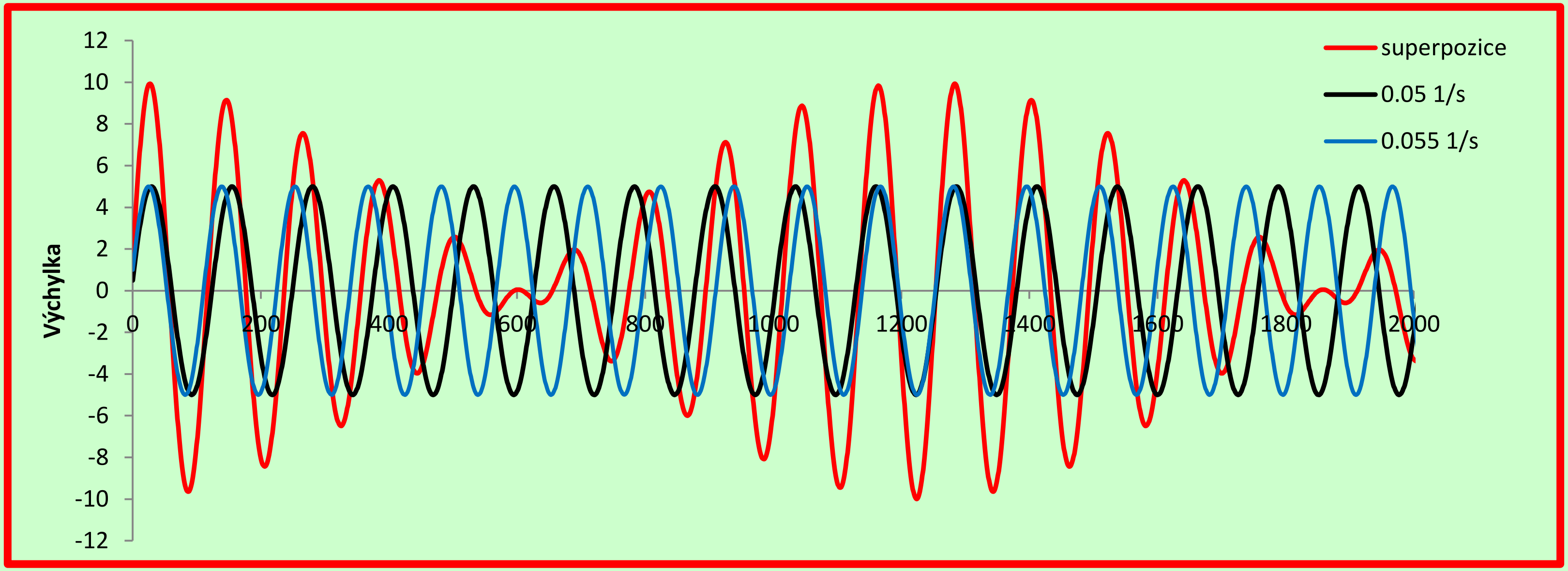
$$u_1(t) = A\sin(\omega_1 t + \varphi_1), \qquad u_2(t) = A\sin(\omega_2 t + \varphi_2)$$

$$u(t) = u_1(t) + u_2(t) = A\sin(\omega_1 t + \varphi_1) + A\sin(\omega_2 t + \varphi_2) =$$

$$= \tilde{A}(t)\sin\left[\frac{\omega_1 + \omega_2}{2}t + \frac{\varphi_1 + \varphi_2}{2}\right]$$

Superposition of oscillations

View of beats



$$\tilde{A}(t) = 2A\cos\left[\frac{\omega_1 - \omega_2}{2}t + \frac{\varphi_1 - \varphi_2}{2}\right]$$

$$\omega_m = \frac{1}{2}(\omega_1 - \omega_2), \quad resp. \quad f_m = \frac{1}{2}(f_1 - f_2)$$

Modulation, beats with frequency 2 f_m
from + and - amplitude modulation values

Animation

Superposition of perpendicular oscillations

Lissajous patterns

1. <u>Both frequencies are the same</u>: $\omega_1 = \omega_2 = \omega$

$$\vec{r}(t) = [x_1(t), x_2(t)]$$

$$x_1(t) = A_1 \sin(\omega_1 t + \varphi_1), \quad x_2(t) = A_2 \sin(\omega_2 t + \varphi_2)$$

$$r(t) = |\vec{r}(t)| = \sqrt{x_1^{\,2} + x_2^{\,2}}$$

- movement is restricted in space:

$|x_1| \leq A_1$, $|x_2| \leq A_2$

$$x_1(t) = A_1 \sin(\omega t + \varphi_1), \quad x_2(t) = A_2 \sin(\omega t + \varphi_2)$$

- the trajectory equation is obtained by excluding time from both equations:

$$\left(\frac{x_1}{A_1}\right)^2 + \left(\frac{x_2}{A_2}\right)^2 - 2\frac{x_1 x_2}{A_1 A_2}\cos(\varphi_2 - \varphi_1) - \sin^2(\varphi_2 - \varphi_1) = 0$$

- the ellipse equation

Superposition of perpendicular oscillations

Lissajous patterns

2. <u>Both frequencies are not the same</u>:

A general curve in a rectangle is created: $-A_1 \leq x_1 \leq A_1$, $-A_2 \leq x_2 \leq A_2$

If the frequencies of ω_1, ω_2 are in the ratio of integers n_1, n_2, i.e. the ratio of the two frequencies is a rational number

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}$$ **kde n_1, n_2 = 1, 2, 3, ….**

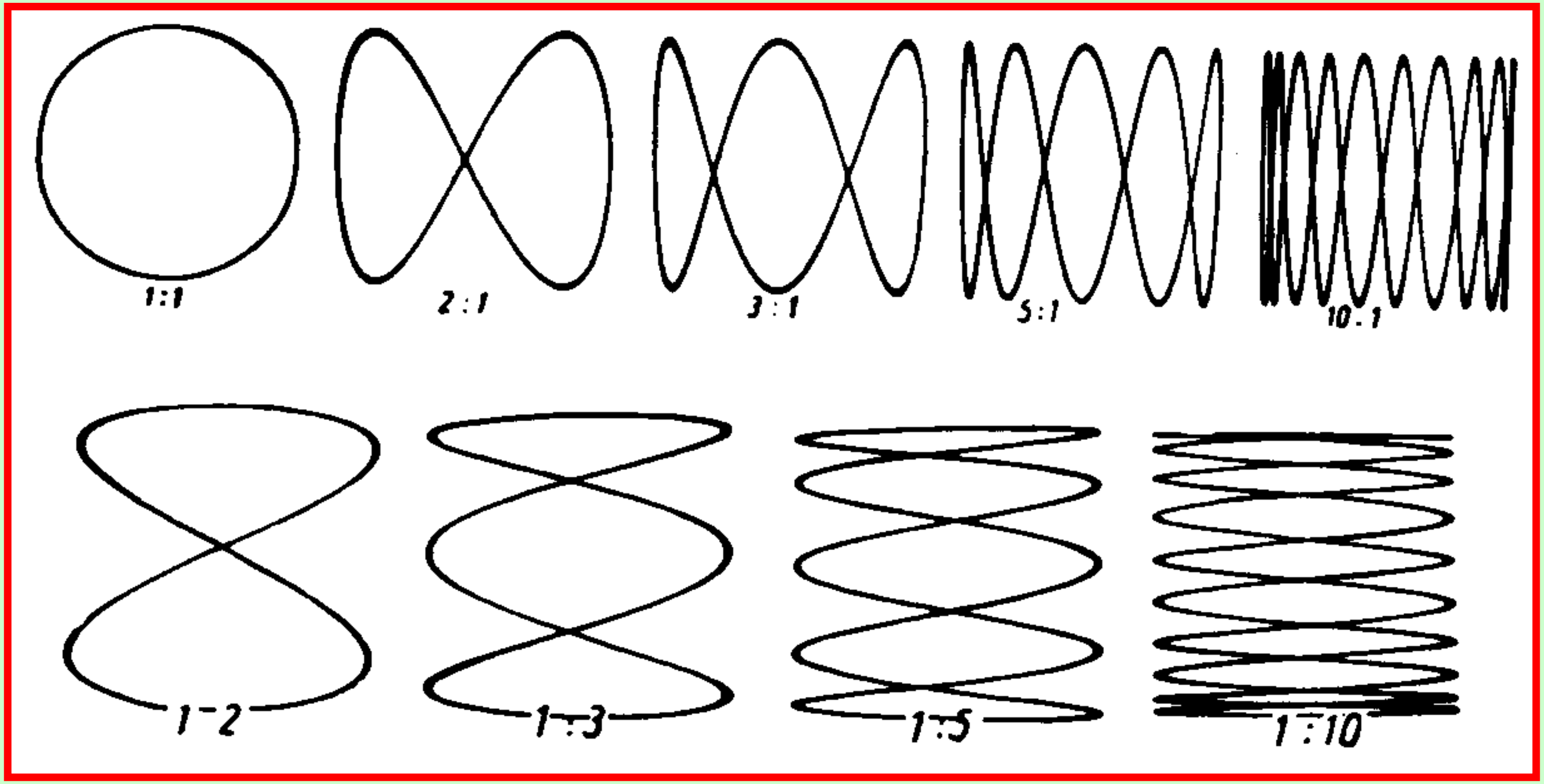
the resulting oscillation has a periodic character. Its period will be equal to the least common multiple of the sub-periods and its frequency to the greatest common divisor of the sub-frequencies.

If the ratio of the frequencies ω_1, ω_2 is not a rational number, the resulting oscillation is not periodic, resulting in an unclosed curve.

Superposition of oscillations in the plane

Different frequencies in integer multiples – Lissajous patterns – curves of the resulting trajectories

b. $x = A \sin \omega_1 t; \quad y = B \sin(\omega_2 t + \varphi)$



Pair of coupled oscillators

Coupling of oscillators 1 – web simulation

Coupling of oscillators 2 – web simulation

Coupling of springs – web simulation + experiment with the model

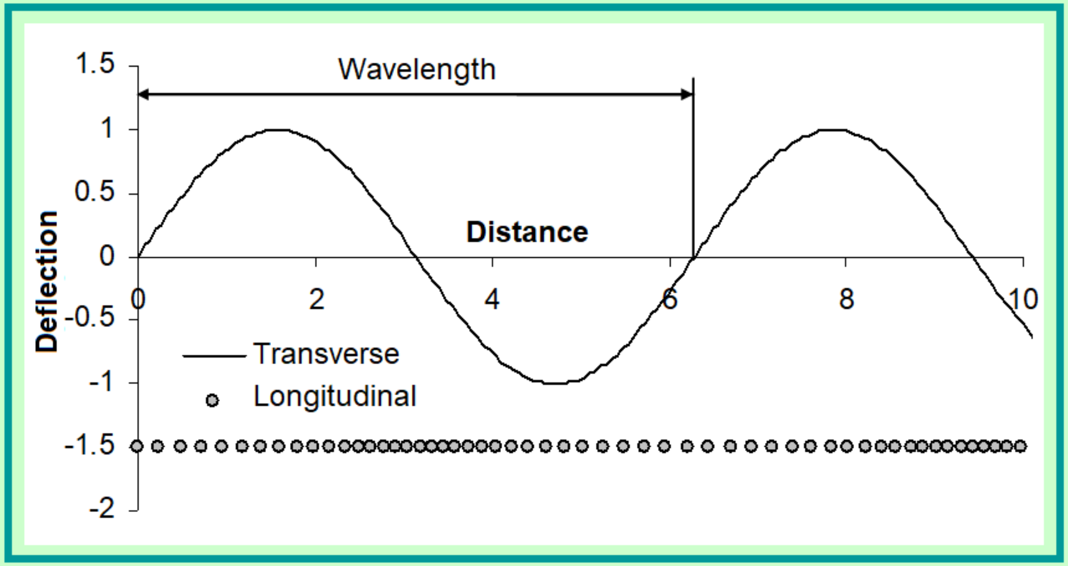
Generation and propagation of successive waves in a material medium

- **in the case of oscillators coupled to each other, the excitation of any of them is associated with the excitation of neighbouring oscillators - the so-called successive wave is formed**
- **the simplest case of wave propagation is along a straight line - longitudinal and transverse**
- **the wave propagation velocity *c* is the phase velocity - in a homogeneous isotropic medium it is constant**
- **in an anisotropic environment, the speed of wave propagation is different in different directions**
- ***λ* – wavelength - the distance travelled by a wave over a period *T***

$$\lambda = cT = \frac{c}{f} = \frac{2\pi c}{\omega}$$

Generation and propagation of successive waves in a material medium

Sequential excitation of oscillators – successive wave



- if there is no change in the plane in which the oscillations take place during the propagation of the transverse traveling wave, the wave is *linearly polarized*

$$\boxed{u(t,x) = A\sin\omega\left(t - \frac{x}{c}\right) = A\sin\omega\left(t - T\frac{x}{\lambda}\right) = A\sin\frac{2\pi}{\lambda}(ct - x)} \quad \frac{2\pi}{\lambda} = k$$ **- angular wave number**

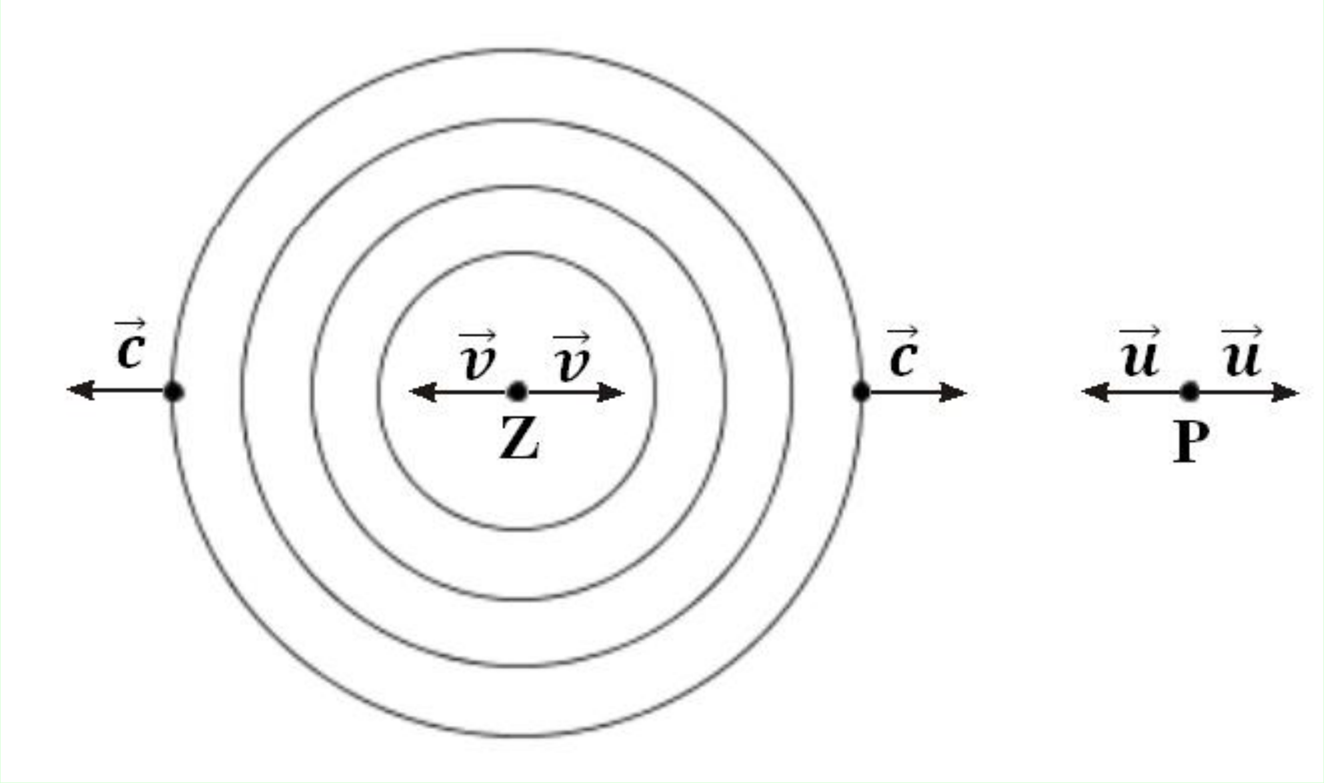
- description of the spatio-temporal wave propagation - the deflection u is a function of time and position

Waves interference

Same principle as in interference of oscillations, but the position must be respected

Doppler effect

- if the source of the waves and its observer move mutually, then in mutual approaching respectively moving away the observer perceives a higher respectively lower frequency of the waves than the natural frequency of the source



[Animation 1]

[Animation 2]

$$f_{\mathrm{P}} = f_{\mathrm{Z}} \frac{c \pm u}{c \mp v}$$

→ approaching of the observer
→ moving away of the observer
→ approaching of the source
→ moving away of the source

Wave equation

Simplified derivation for plane wave

The equation to describe the spatiotemporal propagation of a wave is:

$$u(t,x) = A\sin\omega\left(t - \frac{x}{c}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = -A\omega^2 \sin\omega\left(t - \frac{x}{c}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -A\frac{\omega^2}{c^2} \sin\omega\left(t - \frac{x}{c}\right)$$

Wave equation:

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$$

General solution of the wave equation using functions *f* and *g*:

$$u(x,t) = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right)$$

- waves propagating in the positive (*f*) and negative (*g*) sense along the *x*-axis

Elastic waves

in the material environment

Longitudinal wave propagation in a thin flexible bar

- **Wave propagation is associated with a change in axial displacement Δu to distance Δx,**

i.e. the relative extension is $$\varepsilon = \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$$

- **an elementary section of a rod of length Δx and mass $\Delta m = \rho\,\Delta V = S\rho\,\Delta x$ is set in motion by the difference of forces acting at the ends of the elementary section: $\Delta m.a = \Delta F$, i.e.:**

$$\Delta m \frac{\partial^2 u}{\partial t^2} = S\rho\Delta x \frac{\partial^2 u}{\partial t^2} = \Delta F$$ **(i)** **From Hooke's Law:** $F = S\sigma = SE\dfrac{\partial u}{\partial x}$ **;**

relation (i) can be written: $$S\rho \frac{\partial^2 u}{\partial t^2} = \frac{\Delta F}{\Delta x} = SE \frac{\partial^2 u}{\partial x^2}$$

- **is therefore:** $$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2}}$$ **(ii)**

Elastic waves

in the material environment

Longitudinal wave propagation in a thin flexible bar

- **comparison of equation (ii) with the wave equation** $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ **gives the**

relation: $\boxed{c = \sqrt{\frac{E}{\rho}}}$ **- propagation speed of longitudinal waves in a thin bar**

Elastic waves

Selected formulas for calculating the speed of wave propagation in a material medium

Medium	Type of wave	Formula	Notes
Solid bodies	Transverse	$c = \sqrt{\frac{G}{\rho}}$	***G*** **– modulus of elasticity in shear** ***K*** **– bulk modulus of elasticity** ***μ*** **– Poisson's ratio** ***ρ*** **- density**
Solid bodies	Longitudinal	$c = \sqrt{\frac{3}{\rho}\frac{\mu - 1}{\mu + 3}K}$	
Strings	Transverse	$c = \sqrt{\frac{\sigma}{\rho}}$	***σ*** **– tension in the string** ***ρ*** **- density**
Fluids	Longitudinal	$c = \sqrt{\frac{K}{\rho}}$	***K*** **– bulk modulus of elasticity** ***ρ*** **– density**