Physics of Processes

Generation of standing waves

Generation of standing waves

- interference of opposing progressive waves from x_1, x_2 coordinates with phases:

$$\varphi_1 = -\frac{\omega}{c}(x - x_1), \quad \varphi_2 = \frac{\omega}{c}\left(x - x_2\right)$$

$$x_k = \frac{x_1 + x_2}{2} - \frac{kc\pi}{\omega} = \bar{x} - \frac{k}{2}\lambda$$

- both waves will have the same phase at x_k coordinates, given by the condition: $\varphi_1 = \varphi_2 + k.2\pi$, where k = 1, 2, 3, …

$\bar{x}$ **- mean distance between the both sources**

<u>Coordinates transformation:</u> $x_k = \frac{x_1 + x_2}{2} - \frac{kc\pi}{\omega} = \bar{x} - \frac{k}{2}\lambda$ **- a new origin**

$z = x - \bar{x}$ **, $\delta_v = (x_2 - x_1)/2$**

$$x - x_1 = z + \frac{x_2 - x_1}{2} = z + \delta_v$$

$$x - x_2 = z + \frac{x_1 - x_2}{2} = z - \delta_v$$

$$\varphi_1 = -\frac{\omega}{c}(z + \delta_v), \quad \varphi_2 = \frac{\omega}{c}\left(z - \delta_v\right)$$

$\varphi_1 = -\omega t_s - 2\pi \frac{z}{\lambda}, \quad \varphi_2 = -\omega t_s + 2\pi \frac{z}{\lambda}$ **, where $t_s = \delta_v/c$ - the time of the meeting of the both waves in the middle of the interval (x_1, x_2)**

Generation of standing waves

- the two opposing waves can then be written :

Equal amplitude of opposing waves

$$u_1 = A\sin[\omega(t - t_s) - 2\pi\frac{z}{\lambda}]$$

- both deflections differ in the sign of the phase

$$u_2 = A\sin[\omega(t - t_s) + 2\pi\frac{z}{\lambda}]$$

Superposition of both waves :

$$u = u_1 + u_2 = 2A\cos 2\pi\frac{z}{\lambda}\sin\omega(t - t_s)$$

The resulting equation implies: the interference of two progressive opposing waves produces harmonic oscillations with the same phase but variable amplitude A_v = 2*A* cos(2*πz*/*λ*), which depends on the *z* -coordinate, i.e. the distance of that point from $\bar{x}$.

The largest value 2*A* corresponds to the places with the largest amplitude A_v, i.e. where cos(2*πz*/*λ*) takes values ±1, i.e. for 2*πz*/*λ* = 0, π, 2π, .. Points with this maximum excursion are called ***antinodes***. In the middle between the antinodes lie points where the amplitude of A_v is zero. These points are called ***nodes***.

Animation

Coordinates – antinodes: z = ±kλ/2

Coordinates – nodes: z = ±(2k+1)λ/4

Standing longitudinal waves

In a progressive wave, all points oscillate with the same deflection, but with different phases, which propagate at the phase velocity of the wave. In a standing wave, all points oscillate with the same phase at points a wavelength apart and with opposite phase at points half a wavelength apart. The amplitude value in the case of standing waves is periodically dependent on the position of the point.

Standing waves in the strings

In general: if a traveling wave is reflected at the free or fixed end, it is reflected with the same respectively opposite phase and the resulting standing wave has an antinode respectively node at this end

Strings: threads of different materials fixed and tensioned at both ends

- free oscillations of the string are damped, harmonic and arise from the interference of progressive waves reflecting off the string's fixed edges

Boundary conditions: there are nodes at both ends of the string of length *l*: $n\frac{\lambda_n}{2} = l$

***n* = 1, 2, 3... indicates the fundamental (*n* = 1) respectively higher oscillation frequency (higher harmonic)**

Valid: $\lambda_n = \frac{c}{f_n} = \frac{2\pi c}{\omega_n}$ **, is therefore** $\omega_n = \frac{2\pi c}{\lambda_n} = \frac{\pi n}{l}\sqrt{\frac{\sigma}{\rho}}$ **. Valid:** $c = \sqrt{\frac{\sigma}{\rho}}$ **,**

where σ **- normal tension in the string,** ρ **- string density**

The angular frequencies of the string can also be expressed: $\omega_n = \sqrt{\frac{\sigma}{\rho}}k_\lambda = ck_\lambda$

- relationship between angular frequency and angular wavenumber: $k_\lambda = \frac{2\pi}{\lambda_n}$

- this equation is valid only for integer *n* – it has a discrete (discontinuous) character

Standing waves in thin bars

Elastic solution

$$c = \sqrt{\frac{E}{\rho}}$$

Longitudinal oscillations: **The actual oscillation of the rod is determined by standing harmonic waves, which are generated by the interference of progressive waves in the rod propagating in both principal directions. Their frequency is determined, as in the case of strings, by the arrangement of the standing quarter waves along the rod so as to preserve the nodes at the point of attachment (fixation) of the rod and any antinodes at the free ends of the rod. It is also clear that neither the angular frequency of the natural vibrations nor the speed of their propagation through the rod depends on the cross-section of the rod.**

In the case of *torsional oscillations*, the wave equation form is retained, only the modulus of elasticity in tension E is replaced by the shear modulus G in this equation, while the meaning of the boundary conditions is retained. The biggest change is therefore the difference in the propagation speed of the torsional oscillations. Since $G \approx E/[2(1+\mu)]$, the speed of the torsional waves is $\sqrt{2(1+\mu)}$ - times smaller than the propagation speed of the longitudinal waves (μ denotes the Poisson's ratio).

$$\omega_n = 2\pi K_n \sqrt{\frac{EJ}{l^4 m'}}$$

Transverse oscillations: **E - the modulus of elasticity of the rod, J - the moment of inertia of the cross section of the rod, l – length of the rod, m' - the mass per unit length of the rod, k_λ - the angular wavenumber, $n = l/\lambda$ - the number of wavelengths along the length of the rod, K_n – the constant**

Standing waves in other objects

From the acoustic point of view, the natural vibrations of the plates and membranes are very important, which depend not only on their dimensions and shapes, but also on their fixing points. The positions of the nodes and antinodes on an oscillating plate or membrane can be determined from the distribution of small particles deposited on their surface. Coarser particles (e.g. sand) settle in nodal points or lines and form so-called <u>Chladni patterns</u>. Finer particles (e.g. lycopodium), on the other hand, are deposited at antinodes and form so-called <u>Savart patterns</u>.

- see videos

Of great importance are the so-called spherical (spheroidal) oscillations, i.e. the oscillations of spherical bodies. Among these kmits, the intrinsic kmits of spherical celestial bodies, including the Earth, play a large role.

$$f \approx \sqrt{\frac{Er}{m}}$$

<u>Oscillations of membranes</u>

***E* – modulus of elasticity, *r* – radius, *m* - mass**

Elastic wave energy

- **the propagating wave causes changes in the energy of the environment through which the wave passes**

- **instantaneous deflection of the environment caused by a progressive wave :**

$$u(t,x) = A\sin\omega\left(t - \frac{x}{c}\right)$$

- constraints on the part of the medium with <u>volume</u> Δ*V*, <u>mass</u> Δ*m*, <u>density</u> *ρ* and <u>deflection</u> of the medium from the equilibrium position *u*

$$v(t,x) = \frac{du}{dt} = A\omega\cos\omega\left(t - \frac{x}{c}\right)$$

$$E_k = \frac{1}{2}\Delta m v^2 = \frac{1}{2}\rho\Delta V A^2\omega^2\cos^2\omega\left(t - \frac{x}{c}\right)$$

$$E_p = \int_0^{\Delta l} F\,dl = \int_0^{\Delta l} ES\frac{\Delta l}{l_0}d(\Delta l) = \frac{1}{2}ES\frac{\Delta l^2}{l_0} = \frac{1}{2}ESl_0\left(\frac{\Delta l}{l_0}\right)^2 \quad \frac{\Delta l}{l_0} \approx \frac{du}{dx} = -\frac{A\omega}{c}\cos\omega\left(t - \frac{x}{c}\right)$$

$$E_p = \frac{1}{2}\frac{EA^2\omega^2\Delta V}{c^2}\cos^2\omega\left(t - \frac{x}{c}\right)$$

- the comparison of the relations for E_k and E_p shows that both parts of the energy are in phase (they reach a maximum and a minimum at the same time) - this is the essential difference between the energy of a part of the environment and the energy of a simple oscillating point

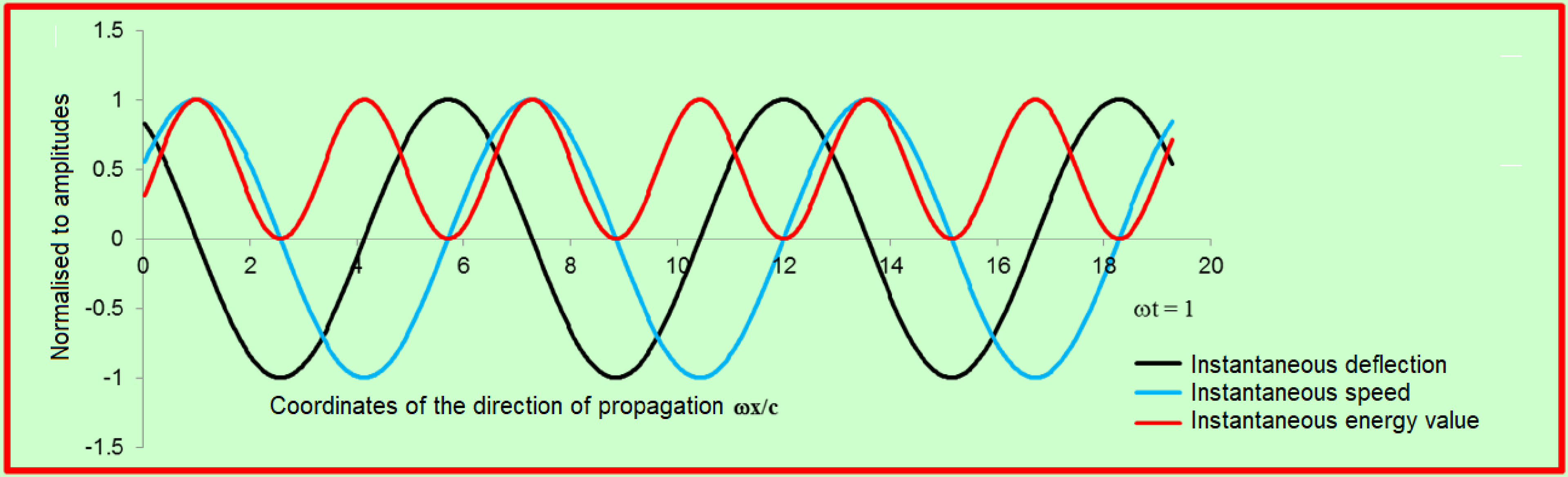
Elastic wave energy

- the total energy of a part of the medium does not remain constant (unlike the total energy of oscillation of a single material point), but it is valid:

$$E_c = E_k + E_p = \frac{1}{2}\left(\rho + \frac{E}{c^2}\right)A^2\omega^2\Delta V \cos^2 \omega\left(t - \frac{x}{c}\right)$$

In an elastic environment $c = \sqrt{\frac{E}{\rho}}$**, then:**

$$E_c = \rho A^2\omega^2\Delta V \cos^2 \omega\left(t - \frac{x}{c}\right)$$



Energy density, wave flux

$$w = \frac{E_c}{\Delta V} = \rho A^2 \omega^2 \cos^2 \omega\left(t - \frac{x}{c}\right)$$

Energy density in an elastic medium

$$\overline{w} = \frac{1}{2}\rho A^2 \omega^2$$

Medium energy density

$$P_e = \overline{w} S c = \frac{S}{2}\rho A^2 \omega^2 c$$

Average wave flow through a plane area *S* (W)

$$\Psi = \frac{P_e}{S} = \frac{1}{2}\rho A^2 \omega^2 c$$

Wave flux density (W.m^{-2})

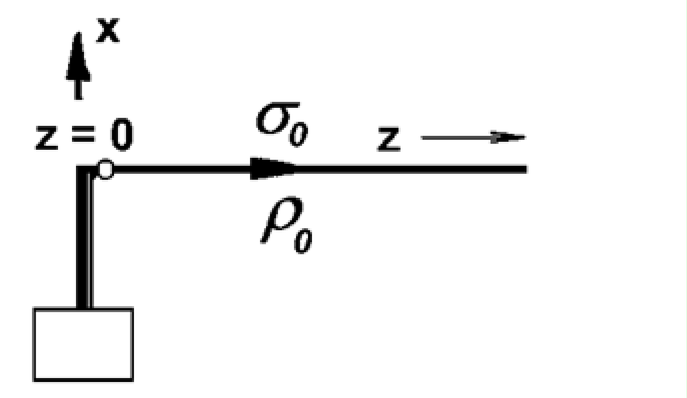
$$\Psi_k = \frac{P_e}{4\pi R^2}$$

Wave flux density of a spherical wave at distance *R* from its source

Animation of sources: simple

another

The formation of a progressive wave in the string



$$c = \sqrt{\frac{\sigma_0}{\rho_0}}$$

c – phase velocity of transverse waves in the string

σ_0 – axial tension in the string

ρ_0 – string density

- the string represents an open system in which a progressive wave propagating in the positive direction *z* is excited:

$$u(t,z) = A\sin(\omega t - k_\lambda z)$$

$k_\lambda = 2\pi/\lambda = \omega/c$ - angular wavenumber

We introduce a quantity characterizing the state of a particular string, a constant independent of the string motion. It is <u>a calibration constant</u> mediating the relationship between the initial transverse wave velocity *c* and the braking stress σ_0. It is called <u>acoustic resistance *Z*</u>:

$$Z = \frac{\sigma_0}{c} = \sigma_0\sqrt{\frac{\rho_0}{\sigma_0}} = \sqrt{\sigma_0\rho_0}$$

Termination and connection of strings

The excited wave is described by the equation: $u_1(t,z) = A\sin(\omega t - k_{\lambda 1} z)$

At the point of transition from the environment of acoustic resistance *Z_1* to the environment of acoustic resistance *Z_2*, a reflected wave is formed, which combines with the original wave.

The total deflection *u* after the addition of the direct and reflected waves for *z* < 0 is then given by:

$$u(t,z) = u_1(t,z) + RA\sin(\omega t + k_{\lambda 1} z),$$
$$u(t,z) = A\sin(\omega t - k_{\lambda 1} z) + RA\sin(\omega t + k_{\lambda 1} z) \quad \text{(a)},$$

where *R* is the reflection coefficient

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

- the equation shows that the reflected wave is phase shifted with respect to the incident wave (difference in signs for the term *$k_{\lambda 1} z$*).

Wave passage through the string interface

The wave generally passes through the interface between the strings. In general, this fact can be expressed by the wave equation in the medium for *z* > 0:

$$u_2(t,z) = TA\sin(\omega t - k_{\lambda 2} z)$$

where *T* is the transmission coefficient,
** *A* is the amplitude of the incident wave,**
** $k_{\lambda 2}$ is the angular wavenumber in the medium for *z* > 0**

Because the deflection of the waves in both parts of the medium must be a continuous function at the interface, the following must hold:

$$u_2(t,0) = u(t,0) = (1+R)u_1(t,0) - \text{viz (a), tj. } TA\sin\omega t = (1+R)A\sin\omega t \Rightarrow \boxed{T = 1+R}$$

Wave passage through the string interface

Different variants of string connection with another environment:

Name	R	T	Z_2/Z_1	Charakteristics
Fixed connection	-1	0	∞	Endless resistance Z_2, standing waves with node for $z = 0$
A simple transition	0	1	1	Equal impedances (not the same environments)
Free end	1	2	0	Zero resistance Z_2, pure standing wave with node for $z = -\lambda/4$ and antinode for $z = 0$

- **if R is larger than -1 and less than 1, then the wave resulting from the superposition of the original and reflected waves <u>is neither a pure standing wave nor a pure travelling wave</u>**

- **such a wave is called a sine wave**

- **each sine wave can be represented by a superposition of two waves either standing or traveling with opposite direction of propagation**

Wave reflection at the interface