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Sensitivity analysis of a pumping test on a well with wellbore storage and skin

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Abstract

This paper reviews the basic concepts of sensitivity analysis and points out their limitations. A case is then made for logarithmic sensitivity. The magnitude of logarithmic sensitivity alone does not determine the accuracy of an aquifer parameter estimate, especially when the relative measurement errors are not uniform throughout space and time. Deterministic parameter correlations and plausible relative errors in parameter estimates are introduced as imperfect measures of information content in measurements. A plausible relative error in the parameter estimate combines the effect of logarithmic sensitivity with that of relative measurement error. Minimizing the plausible relative errors rather than maximizing the corresponding sensitivities should serve as a guide to identifying the measurements most useful for parameter estimation or as candidate measurements for optimal sampling. Furthermore, avoiding among them measurements with high parameter correlations as much as possible may help ensure that the sensitivity matrix $\mathbf{X}$ (or $\mathbf{X}^{\mathrm{T}}\mathbf{X}$) is well-conditioned and, thus, that the parameter estimates are accurate.

The discussed concepts are then applied to a model of a pumping test conducted on a fully penetrating well situated in a confined aquifer. The model accounts for the wellbore storage and an infinitesimal skin. In contrast to the traditional and normalized sensitivities, the logarithmic sensitivities of the drawdown in the pumping well, the drawdown in an observation well, and the wellface flowrate to transmissivity, T, storativity, S, and the skin factor, η, depend on a small number of parameters. They can thus be represented by a single type curve or a family of a relatively few type curves. The plausible relative errors in T, S, and η estimated from wellbore drawdown rapidly decrease during the wellbore storage phase and reach a plateau or slowly decrease outside the wellbore storage phase. The plausible relative errors from the wellface flowrate rapidly decrease during the wellbore storage phase before reaching a minimum (around the time when the wellface flowrate is equal to about half the pumping rate) and then rapidly increase. This means that transient flowmeter test measurements of drawdown and wellface flowrate should not be made during the early times of the wellbore storage phase.

1. Introduction

Dimensionless sensitivity analysis via logarithmic sensitivities has been widely used in sciences because it allows one to compare the sensitivity of one output with respect to one parameter with the sensitivity of another output with respect to yet another parameter. Although very attractive, this approach has been underutilized in hydrology.

The objectives of this paper are: (i) to review the basic concepts of sensitivity analysis and their limitations, (ii) to introduce the concept of a plausible relative error in a parameter estimate, and (iii) to apply logarithmic sensitivity, deterministic correlation, and plausible relative error to a model of a pumping test conducted on a fully penetrating well, with wellbore storage and infinitesimal skin, situated in a confined aquifer.

2. General sensitivity and error analysis

Most engineering, physical, chemical, and biological systems can be viewed as input–output models that relate the output information to the appropriate input parameters. If the model parameters were known perfectly, the output could be calculated. Since this is rarely the case, a question arises: How do the errors in the input parameters influence the outputs, or, in particular, does a small input perturbation cause a large change in the output? Sensitivity analysis was devised to study such questions and, in general, to study the influence of uncertainties in the input data on the outputs.

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Stemming from engineering and applied mathematics, sensitivity analysis is now being widely applied in all sciences. Indeed, a "sensitivity analysis" key word search of the Science Citation Index data base yields well above 500 journal articles published in 1999 and 1998 alone and over 120 articles published in major hydrologic journals in the years 1992–1999. Therefore, only a selective literature review directly relevant to the topic is presented.

The focus here is on pumping tests conducted in homogeneous aquifers – all system parameters are therefore treated as constants rather than functions of space and time, and thus, there is no need for using functional sensitivities (equivalent to Frechet derivatives) [23,27,37–39,51].

2.1. Basic concepts of sensitivity analysis

A number of books [4,8,15,21,24,26,72] and papers ([16,45,63,69,73,74] and many others) present the principles of sensitivity analysis and different approaches to calculating sensitivities. All these principles stem from the concept of a differential.

Consider the evolution of a system whose output $\mathcal{O}(x, t; \mathcal{P}_1, \ldots, \mathcal{P}_n)$ depends on spatial location x, time t and n input parameters $\mathcal{P}_i$. The output, for example, can be the pumping well drawdown, observation well drawdown, wellface flowrate, or other measurable quantity, whereas the parameters may include transmissivity (or hydraulic conductivity) and storativity (or specific storage) of the aquifer or different zones in it as well as the skin factor, leakage factor, etc. The total differential of the output, as a function of spatial location x and time t,

$$\mathrm{d}\mathcal{O} = \sum_{i=1}^{n} \frac{\partial \mathcal{O}}{\partial \mathcal{P}_i} \mathrm{d}\mathcal{P}_i, \tag{1}$$

gives rise to the concept of *traditional sensitivity* (or sensitivity coefficient), understood as the derivative of the output with respect to an input parameter

$$\frac{\partial \mathcal{O}}{\partial \mathcal{P}_i}. \tag{2}$$

When this sensitivity is large in magnitude, the output may be significantly affected even by a small perturbation in the parameter $\mathcal{P}_i$.

The reader should note that traditional sensitivities should not be used to compare the influence of one parameter to the influence of another one when the two have different dimensions. Yet, some authors still make such comparisons. For example, Knopman and Voss [40] state that "sensitivity [of concentration] to dispersion $[\partial c/\partial D]$ is generally an order of magnitude less than sensitivity to velocity $[\partial c/\partial v]\ldots$" This statement may generally be true when one adopts the meter as a unit of length and measures dispersion in meters squared per time and velocity in meters per time, as did these authors. The reverse is true, however, when one adopts the hectometer (100 m) as a unit of length.

Introduced in groundwater literature by McElwee [54], *normalized sensitivity*,

$$\mathcal{P}_i \frac{\partial \mathcal{O}}{\partial \mathcal{P}_i} = \frac{\partial \mathcal{O}}{\partial \mathcal{P}_i / \mathcal{P}_i} = \frac{\partial \mathcal{O}}{\partial \ln(\mathcal{P}_i)}, \tag{3}$$

can be used to compare the influence of one parameter with the influence of another one when the two have different dimensions because, as is apparent from its definition, it measures the influence that the fractional change in the parameter, or its relative error, exerts on the output. (This type of sensitivity is more or less standard in parameter estimation algorithms.)

In turn, the normalized sensitivities should not be used to compare the influence of one or more parameters on different outputs when the outputs are measured in different units (e.g., drawdown and flowrate). However, *logarithmic sensitivity*, defined as

$$\frac{\mathcal{P}_i}{\mathcal{O}} \frac{\partial \mathcal{O}}{\partial \mathcal{P}_i} = \frac{\partial \mathcal{O}/\mathcal{O}}{\partial \mathcal{P}_i / \mathcal{P}_i} = \frac{\partial \ln(\mathcal{O})}{\partial \ln(\mathcal{P}_i)}, \tag{4}$$

can do just that, because it measures the influence that the fractional change in the parameter, or its relative error, has on the fractional change in the output, or its relative error.

Note that it is the first form in (4) that defines the logarithmic sensitivity. It is valid for both positive and negative outputs and parameters. The last form in (4) is obviously less general and is equivalent to the first one only for positive outputs and positive parameters. Yet, it is the last form that gives this sensitivity its name. Logarithmic sensitivity may not be suitable when the output function reaches zero; in such a case normalized or traditional sensitivity might be advantageous. Thus, one should not argue that one type of sensitivity is generally better than another.

2.2. Approaches to calculating sensitivities

Four approaches to calculating sensitivities are most often used: perturbation, analytical, direct, and adjoint methods.

The perturbation approach is the simplest. It approximates the traditional sensitivity of an output variable to a given parameter by a differential quotient of the output perturbation, calculated as the difference between the output with the parameter perturbed and the output with all parameters unperturbed, to the parameter perturbation. This sensitivity can be appropriately scaled to obtain other types of sensitivities. Establishing the right size of the parameter perturbation may not be trivial. The perturbation approach could be prohibitively expensive when sensitivities to numerous parameters are to be calculated or when their evolution in space and/or time is to be studied.

The analytical approach is useful when an analytic representation of a model is available and the sensitivity of an output variable to a given parameter can be explicitly calculated.

The direct approach relies on differentiating the system of equations describing the model with respect to the parameters. The resulting systems of sensitivity equations have a similar structure to the original one and are solved numerically in conjunction with it. The direct method is particularly efficient when the original nonlinear equations are solved via the Newton–Raphson technique. This approach was used by Cho et al. [17], Kabala and Milly [37–39], Reuven et al. [64], and others.

The adjoint method for calculating sensitivities is a popular alternative to the direct method. The system and its sensitivities are described by the governing equation and its adjoint. By solving both numerically, one can extract the sensitivities. The direct and adjoint methods are discussed by Cacuci et al. [14] and Cacuci [13]. The latter provides guidelines for choosing between the two. Typical applications of the adjoint method are provided by Sykes et al. [69], Yeh and Zhang [78], Yeh [79] and others.

In this paper, we use the semi-analytic approach, a variant of the analytical approach, to calculate sensitivities. The model and its sensitivities are all calculated in the Laplace domain and are then numerically inverted to the time domain.

2.3. The role of sensitivities in parameter estimation

Once the n-parameter model $\mathcal{O}(x,t;\mathcal{P}_1,\ldots,\mathcal{P}_n)$, discussed earlier, is established to represent a considered system, the values of its parameters may need to be found. Typically, the system is stressed and its output measured at $m \geqslant n$ different points. Let

$$\begin{aligned} y_i &= \mathcal{O}(x_i,t_i;\mathcal{P}_1,\ldots,\mathcal{P}_n) + \epsilon_i \\ &= \mathcal{O}_i(\mathcal{P}_1,\ldots,\mathcal{P}_n) + \epsilon_i, \quad 1 \leqslant i \leqslant m \end{aligned} \tag{5}$$

be the ith output measurement at the measurement point (x_i,t_i), $\mathcal{O}_i$ be the value of the model at this point, and ϵ_i be the ith (unknown) measurement error.

A number of methods exist to obtain unbiased, minimum-variance, maximum-likelihood, or other estimates of the parameters from these measurements, especially when the statistical nature of the measurement errors is known or could be deduced. In addition, many related approaches are available to optimally select (in some sense) the m measurement points [22,66].

When the measurement errors are Gaussian or of unknown distribution, however, one commonly used way to estimate the system parameters is to minimize the sum-of-squares *objective function* (or sum of squared errors),

$$E = \sum_{i=1}^{m} [\mathcal{O}_i(\mathcal{P}_1,\ldots,\mathcal{P}_n) - y_i]^2. \tag{6}$$

Let us consider first the linear system and then generalize the results to a nonlinear system.

2.3.1. Special case: linear system

For a system linear in its parameters, (5) reduces to

$$y_i = \sum_{j=1}^{n} \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_j} \mathcal{P}_j + \epsilon_i, \quad 1 \leqslant i \leqslant m \tag{7}$$

with constant traditional sensitivities, $\partial\mathcal{O}_i/\partial\mathcal{P}_j$, and (6) becomes

$$E = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_j} \mathcal{P}_j - y_i \right]^2. \tag{8}$$

The objective function (8) obviously has one minimum, which can be found by solving the system of n equations,

$$\frac{\partial E}{\partial \mathcal{P}_k} = \sum_{i=1}^{m} 2 \left[\sum_{j=1}^{n} \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_j} \mathcal{P}_j - y_i \right] \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_k} = 0,$$

i.e., by solving

$$\sum_{j=1}^{n} \mathcal{P}_j \sum_{i=1}^{m} \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_j} \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_k} = \sum_{i=1}^{m} y_i \frac{\partial \mathcal{O}_i}{\partial \mathcal{P}_k}, \quad 1 \leqslant k \leqslant n \tag{9}$$

for the $\mathcal{P}$s.

Alternatively, the linear system (7) can be written approximately (without the error terms) as

$$\mathbf{XP} = \mathbf{Y}, \tag{10}$$

where

$$\mathbf{X} = [\partial\mathcal{O}_i/\partial\mathcal{P}_j] \tag{11}$$

is a rectangular $m \times n$ sensitivity matrix, also known as a Jacobian, $\mathbf{P} = \{\mathcal{P}_1,\ldots,\mathcal{P}_n\}^{\mathrm{T}}$ is the column vector of system parameters, and $\mathbf{Y} = \{y_1,\ldots,y_m\}^{\mathrm{T}}$ is the column vector of output measurements. Multiplication of (10) by $\mathbf{X}^{\mathrm{T}}$, the transpose of $\mathbf{X}$, leads to

$$\mathbf{X}^{\mathrm{T}}\mathbf{XP} = \mathbf{X}^{\mathrm{T}}\mathbf{Y}, \tag{12}$$

which is the matrix form of (9). Since $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is a square $n \times n$ matrix, (12) can be formally solved for the parameter vector, and we thus arrive at the well-known result [22,66]

$$\mathbf{P} = [(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}]\mathbf{Y}. \tag{13}$$

The term in the brackets is known as a *generalized inverse* of $\mathbf{X}$; for a square matrix, it is identical to a regular inverse. Since (12) and (9) are the same, (13) is also a solution of the latter and thus minimizes the objective function (8).

The result (13) was obtained without invoking any assumption about the stochastic nature of the measurement errors, ϵ_i. It is well known, however, that if the measurement errors are Gaussian, (13) is an unbiased estimator, and if they are Gaussian and uncorrelated, (13) is furthermore the minimum-variance estimator [22,66]. The use of the least-squares method, as described above, to obtain parameter estimates implies the expectation that the measurement errors are Gaussian and uncorrelated. Variants of the least-squares method also exist for correlated measurements [22,66].

What if we provide different weights to different measurements? It follows from

$$E = \sum_{i=1}^{m} \alpha_i^2 \left[\sum_{j=1}^{n} \frac{\partial \mathscr{O}_i}{\partial \mathscr{P}_j} \mathscr{P}_j - y_i \right]^2 = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \left(\alpha_i \frac{\partial \mathscr{O}_i}{\partial \mathscr{P}_j} \right) \mathscr{P}_j - \alpha_i y_i \right]^2 \tag{14}$$

and the described relation of (8) to (10) that introducing a weight α_i^2 in the objective function, (14), is equivalent to rescaling the corresponding measurement equation in (10), or in (5) by α_i or, more precisely, is equivalent to rescaling by α_i the ith row in the sensitivity matrix $\mathbf{X}$ (correspondingly, the ith column in $\mathbf{X}^\mathrm{T}$) and the ith measurement in $\mathbf{Y}$.

We note in passing that, for the system with more than one parameter, the measurement equations embedded in (10), or in (5), cannot generally be rescaled for the sensitivity matrix $\mathbf{X}$ to contain the normalized or logarithmic sensitivities, (3) or (4) – this would require the weights to be j-dependent ($\alpha_i = \mathscr{P}_j$ or $\alpha_i = \mathscr{P}_j/\mathscr{O}_i$), which is impossible. Thus, the parameter estimates are usually obtained from (13), with $\mathbf{X}$ being the matrix of *traditional sensitivities,* (2), possibly rescaled.

2.3.2. General case: nonlinear system

Given a sufficiently close approximation, $\mathbf{P}$, of the true, yet unknown, parameter vector, $\mathbf{P}^*$, a system nonlinear in its parameters, (5), can be linearized (in vector form) as

$$\mathbf{X}(\mathbf{P}^* - \mathbf{P}) = \mathbf{Y} - \mathbf{O}, \tag{15}$$

where $\mathbf{O} = \{\mathscr{O}_1, \ldots, \mathscr{O}_n\}^\mathrm{T}$ and the sensitivity matrix $\mathbf{X} = [\partial \mathscr{O}_i / \partial \mathscr{P}_j]$ are both evaluated at $\mathbf{P}$. Since (15) is analogous to (10), the solution for the correction, analogous to (13), is

$$\mathbf{P}^* - \mathbf{P} = [(\mathbf{X}^\mathrm{T}\mathbf{X})^{-1}\mathbf{X}^\mathrm{T}](\mathbf{Y} - \mathbf{O}). \tag{16}$$

We note in passing that, when a close approximation of $\mathbf{P}^*$ is not available, which is typically the case, the linearization (15), or its variant, could be applied iteratively. A number of different methods exist to find nonlinear least-squares estimates [22,66]. In comparison to a linear system, however, all of them share a major complication: as opposed to the linear objective function, (8), the nonlinear objective function, (6), may have multiple minima. Thus, an iterative solution based on (15) or other approaches may lead to its local rather than global minimum [22,66].

It is important to note from (13) and (16) that, whether the system is linear or nonlinear, the quality of the least-squares parameter estimates is controlled by the nature of the sensitivity matrix $\mathbf{X}$, (11), or, more precisely, by the nature of $\mathbf{X}^\mathrm{T}\mathbf{X}$. In fact, the existence of the least-squares parameter estimates is predicated on the existence of the inverse of $\mathbf{X}^\mathrm{T}\mathbf{X}$; when this matrix is singular or ill-conditioned, the solution does not exist or is highly inaccurate, respectively.

2.4. Perspectives of sensitivity analysis

There are two different perspectives of sensitivity analysis present in the literature: the *parameter-estimation perspective* and the the *sensitivity-only perspective*.

The *parameter-estimation perspective* focuses on parameter estimation via solving an appropriate inverse problem, which involves calculation of sensitivities, and/or formulating an optimal sampling design. In the process, a number of simplifying assumptions about the statistical structure of the measurement errors (independent, Gaussian, constant variance, etc.) are invoked, and parameter correlation is accounted for. In this perspective, the main question is: *Given system measurements, what are the parameter estimates and their variances*? In subsurface hydrology, the parameter-estimation perspective was adopted by Butler and Liu [11,12], Cob et al. [18], Knopman and Voss [40,41,43], Knopman et al. [44], McElwee [54], McElwee et al. [55,56], McElwee and Yukler [57], and others.

In the *sensitivity-only perspective*, the focus is on the inherent nature of sensitivities, and the study does not involve parameter estimation. In this perspective, the main questions (directly related to each other, as will be seen shortly) are: *How much does a small, deterministic perturbation of a parameter (input) affect the system (output)*? or *What is the sensitivity of the system to its parameters*? The deterministic perturbation is usually given as a typical measurement error, specified for an instrument (such as a pressure transducer or a flowmeter) by the manufacturer. Statistical characterization of the measurement errors is neither given, nor assumed, nor considered. Ideally, the nature of the whole sensitivity matrix, $\mathbf{X}$, given in (11), or better still, the nature of $\mathbf{X}^\mathrm{T}\mathbf{X}$, should be studied. Practically, however, studying the sensitivities and the deterministic parameter correlations provides useful insights into the nature of these matrices.

The sensitivity-only perspective was adopted in books by Chatterjee and Hadi [15] and Deif [21] and in papers by Chen et al. [16], Cho et al. [17], Koda et al. [45], Rabitz [63], Reuven et al. [64], Ungureanu et al. [73], Vajda and Rabitz [74], and many others. In subsurface hydrology it was adopted by Kabala and Milly [37–39] and to a large degree by Cob et al. [18], McElwee et al. [56], and others. This perspective can provide qualitative information about which model parameters are likely to be accurately estimated and which measurement points (in space or time) should be considered as candidates for the sampling design.

The *sensitivity-only perspective* is adopted for this paper.

3. Selected applications of sensitivity analysis in hydrology

Sensitivity analysis has been applied in solving inverse problems and estimating model parameters, in finding optimal sampling designs, as well as in studying sensitivities for their own sake. The large-perturbation approach and traditional sensitivities have been widely used in hydrologic problems, whereas normalized sensitivities and, especially, logarithmic sensitivities are used less often.

3.1. Sensitivity analysis in inverse modeling and sampling design

In the *parameter-estimation perspective*, sensitivities are used to define an appropriate Jacobian and to solve an inverse problem, i.e., to estimate model parameters from available measurements. This approach has been used by Hughson and Yeh [32,33], Kool and Parker [47], Kool et al. [48,49], Parker et al. [62], Shah et al. [67], Yeh and Zhang [78], Yeh [79] and many others.

Sensitivity analysis can also be used quantitatively in finding an optimal sampling design. A number of optimality criteria exist that are believed to be important for experimental designs [66]. Each criterion is labeled in the literature by a single letter; the criteria are known jointly as "alphabet optimality". For example, D-optimality minimizes the volume of the confidence region of the regression parameters. G-optimality, also known as a minimax criterion, minimizes the maximum output variance over all inputs in the experimental region, whereas A-optimality minimizes the average variance of the regression parameters. Other alphabet optimality criteria exist (E, I, and J) and are discussed by Ryan [66]. They are generally not equivalent to one another. Although the D-optimal sampling design is most widely used, it is not necessarily an equileverage design (or, equivalently, G-optimal design [66]), i.e., not all its design points exert equal influence on the determination of regression parameters, as one would wish. However, a sampling design may be both D- and G-optimal [66].

Some of the alphabet optimality criteria have already been applied to hydrology. For example, Knopman and Voss [40,41,43] and Knopman et al. [44] explored applications of D-optimality in sampling design of contaminant transport in groundwater, whereas Knopman and Voss [43] commented on the A-optimality of such designs. Yet none of the existing hydrologic sampling designs seems to be both D- and G-optimal.

One should recognize that any algorithm for selecting an optimal sampling design will produce one that is optimal only on the finite number of considered candidate measurement points and, thus, a design that is not necessarily globally optimal [66]. One should also recognize that the alphabet optimality criteria, which involve sensitivity matrices, require invoking an assumption about the statistical structure of the measurement errors. Usually, it is assumed that these errors are independent, of equal and constant variance, and either normally or log-normally distributed [66]. Yet none of these assumptions needs to be true.

3.2. Large-perturbation approach – a study of output variation due to input variation

A number of researchers do not use the formalism of sensitivity analysis and prefer to directly vary the parameters within their ranges and calculate the corresponding changes in the output. Although this approach does not suffer from the small-perturbation assumption of the first-order sensitivity analysis, (2)–(4), it does not provide the same level of local generality as do the definitions (2)–(4). The large-perturbation approach has been used by many researchers. A few examples follow.

Griggs and Peterson [29] applied this approach to the atoll subsurface hydrology model to find out that the modeled depth to the 50% salinity contour is most sensitive to permeability, and the transition-zone thickness is most sensitive to transverse dispersivity. Corapcioglu and Choi [19] found that the aqueous phase colloid concentration is quite sensitive to changes in the rate coefficient of colloidal deposition on the solid matrix in colloid transport in unsaturated porous media. Abboud and Corapcioglu [1,2] studied the effect of mud penetration on borehole skin properties. Minsker and Shoemaker [58] quantified the economic and environmental effects of uncertainty in biological parameter values in optimal in-situ bioremediation design. Lahm et al. [50] found that increasing dispersivity causes an increase in the variable-density effects in their two-dimensional numerical transport model of brine displacement by infiltrating meteoric water. Ruan and Illangasekare [65] revealed that the interaction of macropore flow and overland flow is significant in their numerical model coupling overland flow and infiltration into the vadose zone with macropores. Bolster et al. [7]

conducted miscible displacement experiments in sand columns to determine the variability of bacterial deposition scale within aquifer sediments, and used the large-perturbation approach sensitivity analysis to study the effects of an influent suspension with two subpopulations of bacteria on the decrease of deposited bacteria with flow path length. Also, Boateng and Cawlfield [5] applied the large-perturbation approach sensitivity analysis to two-dimensional contaminant transport in the unsaturated zone.

3.3. Applications of traditional sensitivities

Traditional sensitivities, (2), are the most often applied tools of sensitivity analysis. Numerous applications are available. Selected examples follow.

Vemuri et al. [76] applied traditional sensitivities to system identification and pointed out its potential in hydrologic research. Similarly, McCuen [53] stressed the usefulness of sensitivity analysis for dealing with complex water resources systems. McElwee and Yulker [57] studied the sensitivity of groundwater models to variations in transmissivity and storage. Cob et al. [18] extended this analysis to leaky aquifers. Knopman and Voss [40] studied the behavior of sensitivities and their implications for parameter estimation in the one-dimensional convection–dispersion equation. Butler and Liu [11] demonstrated the importance of temporal and spatial placement of observations for interpreting pumping tests conducted in linear infinite strip aquifers. They found that changes in drawdown are sensitive to the transmissivity and storativity of the strip only for a limited time and extremely limited time, respectively.

Sim and Chrysikopoulos [68] developed a one-dimensional porous media model for virus transport in saturated porous media. Their sensitivity analysis revealed that the estimation of pseudo first-order inactivation rate coefficients from field observations requires data collection near the source during the initial stages of virus transport. Vasco and Datta-Gupta [75] developed a tracer tomography technique that involves calculating tracer concentration sensitivities to porosity, permeability, and pressure gradient and used them for inversion of tracer data. They employed a numerical-perturbation approach and the asymptotic semianalytic approach along a streamline to calculate traditional sensitivities of concentration to permeability and concentration to porosity. Belitz and Dripps [3] found that the response of the slugged well is mostly sensitive to radial hydraulic conductivity, and less sensitive to anisotropy and the conductivity of the borehole skin, and nearly insensitive to specific storage, whereas the responses of the observation wells are sensitive to all four parameters.

3.4. Applications of normalized sensitivity

Normalized sensitivity, (3), has been successfully applied in analyzing well tests. McElwee [54] reviewed the principles of sensitivity analysis and approaches to calculate traditional and normalized sensitivities, gave examples of traditional sensitivities for the Theis and Hantush models and discussed the boundary effects on sensitivities as well as the role of sensitivities in the estimation of aquifer parameters and associated confidence intervals. Bohling and McElwee [6] coded sensitivity analysis of pumping tests. Their program aids in the design and analysis of pumping tests and slug tests. Following up the research of Butler and Liu [11] for strip aquifers, Butler and Liu [12] found analogous results for pumping tests in a nonuniform aquifer conceptualized as a uniform matrix with a disk of anomalous properties placed at an arbitrary location with respect to the pumping well. In particular, they found that changes in drawdown are sensitive to the hydraulic properties of the disk for a limited time only and that, at observation wells located at moderate to large distances from the pumping well, the effect of spatial variations in flow properties is negligible. They further confirmed that constant rate pumping tests are not effective in characterizing lateral variations in flow properties. McElwee et al. [55] studied slug tests and found that their sensitivity to storativity is much lower than that to transmissivity and that the two parameters are highly correlated. McElwee et al. [56] demonstrated that the use of one or more observation wells can vastly improve the parameter estimates, particularly the storativity. Jiao and Lerner [34] used normalized sensitivities to determine how to zone the parameters in ground water flow models. Jiao and Zheng [35] found that an upstream observation well can produce information on storativity both upstream and downstream, but it can produce little information on transmissivity downstream.

3.5. Applications of logarithmic sensitivity

Dimensionless in its nature, logarithmic sensitivity has been widely used in sciences. For example, Zhou and Stone [80] utilized it in a climate model study to describe the sensitivity of eddy heat flux to temperature gradient, Chen et al. [16] used it to describe ozone concentration sensitivity to changes in chemical reaction rates, Filippetti et al. [25] used it in hardness theory to evaluate the transferability of semi-local pseudopotentials, and Okuyama et al. [60] used it to describe the change in an interface barrier height (tunnel current) to hydrogen gas. It has also been widely used in economics ([9,10] and many others).

Surprisingly, the powerful concept of logarithmic sensitivity, (4), has not yet been widely used in hydrology. Liou and Yeh [52] used it (but plotted transformed logarithmic sensitivities only) to evaluate risk in one-dimensional groundwater transport models with uncertain groundwater velocity and longitudinal dispersivity. They found that the mean longitudinal dispersivity is the most sensitive parameter and the variance of longitudinal dispersivity is generally the least sensitive. Munoz-Carpena et al. [59] employed logarithmic sensitivity in modeling hydrology and sediment transport in vegetative filter strips – they plotted percent change in outputs (runoff volume, peak runoff rate, and delay time) to the percent change in inputs (saturated hydraulic conductivity, initial moisture content, media spacing, etc.).

Note in passing that, although Butler and Liu [12] and Jiao and Zheng [35] applied the normalized sensitivity concept to the nondimensionalized drawdown, their dimensionless sensitivity differs significantly in its interpretation from logarithmic sensitivity.

4. A case for logarithmic sensitivities and a new measure of information content

The main drawback of the normalized sensitivity, (3), and traditional sensitivity, (2), is their dimensional character. As mentioned earlier, neither concept is applicable when two different outputs are measured, such as wellbore drawdown and wellface flowrate in the flowmeter test. The dimensionless logarithmic sensitivity, (4), however, is useful in such cases. In fact, it has an appealing interpretation.

Consider again an evolution of a system $\mathcal{O}(x, t; \mathcal{P}_1, \ldots, \mathcal{P}_n)$. For a nonzero output, its total differential as a function of x and t, (1), can be rearranged as

$$\frac{\mathrm{d}\mathcal{O}}{\mathcal{O}} = \sum_{i=1}^{n} \left(\frac{\mathcal{P}_i}{\mathcal{O}} \frac{\partial \mathcal{O}}{\partial \mathcal{P}_i} \right) \frac{\mathrm{d}\mathcal{P}_i}{\mathcal{P}_i}, \tag{17}$$

which naturally gives rise to logarithmic sensitivities (4). Each logarithmic sensitivity can thus be interpreted as a transfer coefficient between the relative error in the input parameter and the relative error this input parameter alone induces in the output.

4.1. Relative measurement error in the output

Assume now that we measure the output with some typical measurement error, for example, an absolute maximum measurement error $\Delta\mathcal{O}$, as often specified for instruments (such as pressure transducers or flowmeters). This error may generally vary with the output $\mathcal{O}$. The corresponding relative measurement error in the output is

$$\frac{\Delta\mathcal{O}}{\mathcal{O}} = f(\mathcal{O}), \tag{18}$$

where $f(\mathcal{O})$ is the calibration curve of the instrument used in the measurements. This curve should be provided by the instrument manufacturer or could be obtained by calibrating the instrument. Given the calibration curve, (17) can be written as

$$f(\mathcal{O}) = \sum_{i=1}^{n} \left(\frac{\mathcal{P}_i}{\mathcal{O}} \frac{\partial \mathcal{O}}{\partial \mathcal{P}_i} \right) \frac{\mathrm{d}\mathcal{P}_i}{\mathcal{P}_i}. \tag{19}$$

4.2. The maximum sensitivity principle for selecting measurements

Knopman and Voss [40], who performed sensitivity analysis of the convection–dispersion equation, and McElwee [54] demonstrated and emphasized the usefulness of sensitivities in estimating variance, confidence intervals, and/or confidence regions. In addition, they formulated what amounts to the *maximum sensitivity principle for selecting measurement points* in time and location. Indeed, Knopman and Voss [40] listed among "several principles [that] emerge from [their] analysis":

> Information about a physical parameter may be most accurately gained at points in space and time with a high sensitivity to the parameter. Taking observations at [these points] tends to yield relatively low variance in the estimate of the parameter...

McElwee [54] argued for maximizing model sensitivities as a way of improving parameter estimation and provided the corresponding *maximum sensitivity principle* for pumping tests:

> Some general guidelines can be given for increasing the model sensitivity, leading to more accurate parameter estimation. For maximum sensitivity, the measurements of head should occur at locations and times where the sensitivity coefficients are near their maximum values.

The "weak form" of the maximum sensitivity principle for selecting measurement points was given by an anonymous reviewer in the following pithy maxim:

> Measurement points with high sensitivities can never be bad.

Following the maximum sensitivity principle is supposed to ensure optimal or nearly optimal parameter estimation. Knopman and Voss [41] relied on it when they stated that "the peak of the sensitivity curve is the most desirable point to make an observation for the accurate observation of velocity". Similarly, Sim and Chrysikopoulos [68] relied on the maximum sensitivity principle when they concluded that "the virus transport data collected in the vicinity of the source of contamination at early time are most reliable for estimation of inactivation rate coefficients''. Also, McElwee et al. [55,56] claimed that ''The best estimates for T and S are obtained by ... and sampling at points of maximum sensitivity''. The maximum sensitivity principle seems to have been generally accepted by these and other researchers ([35] and others).

As follows from (19), however, the maximum sensitivity principle is true only under the rather unrealistic assumption that the relative error in the output measurements is the same for all output values. Indeed, selecting measurements at times and locations around the maximum sensitivity cannot in itself ensure the most accurate parameter estimation, especially, when the measurements at these times or locations are biased by significantly larger errors than at other times or locations. A simple counterexample to the maximum sensitivity principle is a system whose output has the highest sensitivity at times when (or at spatial locations where) the output measurements are still below the detection limit, i.e., when (or where) they are biased by very large relative errors. The model considered later in this paper provides such counterexamples.

Generally, one should thus focus not on sensitivities alone, but rather consider them together with the relative measurement errors. This is especially the case when statistical characterization of the measurement errors is not available while the typical (average, maximum) error is known. A new information-content measure that combines output sensitivity to a parameter with the relative measurement error in the output may be useful for this purpose.

4.3. Imprecise measures of information content in the system output

As mentioned earlier, the quality of the least-squares parameter estimates depends on the nature of the sensitivity matrix $\mathbf{X}$, (11), and, more generally, on the nature of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$.

One global measure of the parameter estimates' quality could be the *matrix condition number* [28]

$$\mathrm{cond}(\mathbf{X}^{\mathrm{T}}\mathbf{X}) = \|(\mathbf{X}^{\mathrm{T}}\mathbf{X})\| \left\| (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1} \right\|, \tag{20}$$

where $\|\cdot\|$ is the matrix norm, for example, the maximum norm or the Euclidean norm. Multiplication of the norm of a matrix and the norm of its inverse provides a normalizing effect. Thus, only for an ill-conditioned system is the condition number large. The condition number is used, for example, to provide bounds for the relative error in the norm of the solution of a linear system [28, p. 111] and thus could be used in specifying the bounds for the relative error in the norm of $\mathbf{P}$ in (13). Although such bounds provide information about the quality of the whole parameter vector, they say nothing about the quality of its components. Insights about the latter could be gleaned from studying the sensitivities of the parameter vector components.

When the number of measurements is equal to the number of parameters, $n = m$, it suffices to consider only the sensitivity matrix $\mathbf{X}$ and its condition number, cond($\mathbf{X}$). For this case, Stoer and Bulirsch [70, p. 13] observed that large absolute values of *logarithmic sensitivities,* rather than traditional sensitivities, are possible signatures of an ill-conditioned problem. This provides one more reason to study logarithmic sensitivities.

However, the sensitivities alone do not tell the whole story. Their relation to each other and their relation to the relative measurement error, $f(\mathcal{O})$, are also important.

Whenever a number (larger than one) of parameters are estimated simultaneously, their accuracy depends not only on measurement noise and the magnitude of the parameter sensitivities, but also on the *deterministic correlation* (or co-variance) between the parameters. Not to be confused with the statistical correlation of two random variables, deterministic correlation is determined solely by the parameter sensitivities. Indeed, consider how an error $\mathrm{d}\mathscr{P}_j$ in $\mathscr{P}_j$ would affect $\mathscr{P}_i$, given a perfect measurement and perfect knowledge of all the other parameters. From (1) we arrive at the *deterministic correlation* of the parameter $\mathscr{P}_i$ with the parameter $\mathscr{P}_j$ as the ratio of traditional sensitivities, (2),

$$R_{P_iP_j} = -\frac{\mathrm{d}\mathscr{P}_i}{\mathrm{d}\mathscr{P}_j} = \frac{\partial\mathcal{O}/\partial\mathscr{P}_j}{\partial\mathcal{O}/\partial\mathscr{P}_i}. \tag{21}$$

The main drawback of this expression is its dimensionality.

In an analogous manner, from (17) we arrive at the *deterministic logarithmic correlation* of the parameter $\mathscr{P}_i$ with the parameter $\mathscr{P}_j$ as the ratio of the logarithmic sensitivities, (4),

$$R^{\log}_{P_iP_j} = -\frac{\mathrm{d}\mathscr{P}_i/\mathscr{P}_i}{\mathrm{d}\mathscr{P}_j/\mathscr{P}_j} = \frac{(\mathscr{P}_j/\mathcal{O})\partial\mathcal{O}/\partial\mathscr{P}_j}{(\mathscr{P}_i/\mathcal{O})\partial\mathcal{O}/\partial\mathscr{P}_i}. \tag{22}$$

This correlation is dimensionless. Note that $R^{\log}_{P_iP_j}$ is large in magnitude when the logarithmic sensitivity to $\mathscr{P}_j$ is larger in magnitude than the logarithmic sensitivity to $\mathscr{P}_i$. Usually, the absolute values of relative errors and correlations are used.

Now, consider how the relative measurement error translates into the parameter estimation error for the two parameters, given that the correlation $R^{\log}_{P_iP_j}$ is high and knowledge of all the other parameters is perfect. It follows from (22) that a small change in the relative error in $\mathscr{P}_j$ will lead then to a much larger change in magnitude in the relative error in $\mathscr{P}_i$, rendering the estimate of the latter much less reliable. Thus, large correlations of a parameter with other parameters lead to large parameter uncertainties, in spite of high absolute values of sensitivities. In fact, if, for all measurements, a given parameter is highly correlated with one or more other parameters, then its estimate will have large uncertainty. Thus, in selecting the measurement points, we should strive to ensure, if possible, that at least at one of these points each parameter will have low correlations to other parameters.

In an inverse problem, we want to estimate all n input parameters $\mathscr{P}_i$ from the measured evolution record of the output $\mathscr{O}$. That the logarithmic (as well as traditional and normalized) sensitivities cannot tell the whole story about the quality of these parameter estimates is now clearly apparent from (19), which involves not only the sensitivities, but also the relative measurement error $f(\mathscr{O})$.

Therefore, another measure is introduced to account for output sensitivity to a parameter and for the relative measurement error in the output. Relation (19) provides inspiration for it. Consider again an input parameter, $\mathscr{P}_i$, estimated from the evolution of a system output, $\mathscr{O}(x,t;\mathscr{P}_1,\ldots,\mathscr{P}_n)$, measured with the relative error, $f(\mathscr{O})$. The *plausible relative error* in $\mathscr{P}_i$ is

$$\frac{d\mathscr{P}_i}{\mathscr{P}_i} = f(\mathscr{O}) \bigg/ \left(\frac{\mathscr{P}_i}{\mathscr{O}}\frac{\partial \mathscr{O}}{\partial \mathscr{P}_i}\right). \tag{23}$$

As follows from (19) and (23), it can be interpreted as a relative error in the input parameter that one would obtain from the relative measurement error of the output, $f(\mathscr{O})$, given perfect knowledge of all the other parameters.

Plausible relative error provides a measure of information content about the parameter contained in a single output measurement – the larger the plausible relative error in the parameter, the less information about it is contained in the output measurement.

It should be emphasized that the defined plausible relative error in $\mathscr{P}_i$ does not have to be, and generally is not, equal to the actual error in the estimate of $\mathscr{P}_i$. In fact, the actual error in the parameter estimate is based on the temporal record (or spatial record, or both) or its part of the measured outputs, whereas the plausible relative error at time t describes the information content about the parameter contained in one measurement point, $\mathscr{O}(x,t;\mathscr{P}_1,\ldots,\mathscr{P}_n)$, obtained at this very time t.

In the course of this paper, it will be further illustrated that the plausible relative error, as a measure of information content about $\mathscr{P}_i$ contained in $\mathscr{O}$, is more suitable than the traditional, normalized or logarithmic sensitivities to serve as a guide in identifying the useful portions of the measured output record for parameter estimation. In fact, with the earlier cited observation of Stoer and Bulirsch [70, p. 13] that large absolute values of logarithmic sensitivities are possible signs of an ill-conditioned problem, we can replace the earlier discussed maximum sensitivity principle for selecting measurement points with

Heuristic guidelines for measurement selection:

Minimizing the plausible relative errors rather than maximizing the corresponding sensitivities should serve as a guide to identifying the measurements most useful for parameter estimation or as candidate measurements for optimal sampling. Furthermore, avoiding among them as much as possible the measurements with high parameter correlations may help ensure that the sensitivity matrix $\mathbf{X}$ (11), (*or* $\mathbf{X}^T\mathbf{X}$) *is well-conditioned and thus that the parameter estimates are accurate.*

4.4. Fixed absolute measurement error model

It follows from (18) that the absolute measurement error, $\Delta\mathscr{O} = \mathscr{O}f(\mathscr{O})$, is generally a function of the output $\mathscr{O}$ (depending on conditions, such as temperature and pressure, one can have different calibration curves for the same instrument). However, in the absence of the calibration curve, the instrument manufacturer provides a typical or maximum absolute measurement error $\Delta\mathscr{O}_{\max}$. In such a case, one could assume that

$$\Delta\mathscr{O} = \text{const} = \Delta\mathscr{O}_{\max} \tag{24}$$

is independent of the measurement $\mathscr{O}$ and thus define an approximate calibration curve as

$$f(\mathscr{O}) = \frac{\Delta\mathscr{O}_{\max}}{\mathscr{O}}. \tag{25}$$

For this relative measurement error model, the definition of the plausible relative error in the parameter, (23), that is compatible with the measurement error reduces itself to

$$\frac{d\mathscr{P}_i}{\mathscr{P}_i} = \Delta\mathscr{O}_{\max} \bigg/ \left(\mathscr{P}_i\frac{\partial \mathscr{O}}{\partial \mathscr{P}_i}\right). \tag{26}$$

In this paper, (25) and (26) will be mostly used. However, the presented methodology is general and, with (23), it can accommodate an arbitrary form of the calibration curve, $f(\mathscr{O})$.

5. Semi-analytic pumping test model

Consider a fully penetrating well situated in a confined homogeneous aquifer [31] of horizontally infinite extent. The initial boundary value problem (IBVP) for the well response that accounts for wellbore storage and infinitesimal skin is

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}, \tag{27}$$

$$s|_{t=0} = 0, \tag{28}$$

$$s|_{r=\infty} = 0, \tag{29}$$

$$s_{\mathrm{w}} = s|_{r=r_{\mathrm{w}}} - \eta r_{\mathrm{w}} \frac{\partial s}{\partial r}\bigg|_{r=r_{\mathrm{w}}}, \tag{30}$$

$$q = 2\pi r_{\mathrm{w}} T \frac{\partial s}{\partial r}\bigg|_{r=r_{\mathrm{w}}} = -\left(Q - \pi r_{\mathrm{c}}^2 \frac{\mathrm{d}s_{\mathrm{w}}}{\mathrm{d}t}\right), \tag{31}$$

where $s(r,t)$ is the drawdown at the distance r from the pumping well and time t, S aquifer storativity, $T = Kb$ aquifer transmissivity, K aquifer hydraulic conductivity, b aquifer thickness, r_{w} well radius, s_{w} wellbore drawdown, r_{c} radius of the well casing, η skin factor, Q total pumping rate, and q the wellface flowrate.

Note in passing that the prevalent heterogeneities are much more important in contaminant transport than in water flow through the subsurface. For the latter the concept of an "effective porous medium" (i.e., a hypothetical homogeneous medium that provides virtually the same response as the given heterogeneous medium) has been long in use in subsurface hydrology. It is in this sense that the aquifer homogeneity assumption makes the considered model applicable to real aquifers. Although very interesting, the issue of aquifer heterogeneity is beyond the scope of this paper.

Let us introduce the following dimensionless parameters

$$s_{\mathrm{D}} = \frac{s}{r_{\mathrm{w}}}, \tag{32}$$

$$s_{\mathrm{wD}} = \frac{s_{\mathrm{w}}}{r_{\mathrm{w}}}, \tag{33}$$

$$\rho = \frac{r}{r_{\mathrm{w}}}, \tag{34}$$

$$\tau = \frac{Tt}{Sr_{\mathrm{w}}^2}, \tag{35}$$

$$q_{\mathrm{D}} = \frac{q}{Q}, \tag{36}$$

$$\alpha = \frac{r_{\mathrm{w}}^2}{r_{\mathrm{c}}^2} S, \tag{37}$$

$$\gamma = \frac{2\pi r_{\mathrm{w}}}{Q} T. \tag{38}$$

The dimensionless analog of IBVP (27)–(31) is

$$\frac{\partial^2 s_{\mathrm{D}}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial s_{\mathrm{D}}}{\partial \rho} = \frac{\partial s_{\mathrm{D}}}{\partial \tau}, \tag{39}$$

$$s_{\mathrm{D}}|_{\tau=0} = 0, \tag{40}$$

$$s_{\mathrm{D}}|_{\rho=\infty} = 0, \tag{41}$$

$$s_{\mathrm{wD}} = s_{\mathrm{D}}|_{r=r_{\mathrm{w}}} - \eta \frac{\partial s_{\mathrm{D}}}{\partial \rho}\bigg|_{\rho=1}, \tag{42}$$

$$q_{\mathrm{D}} = \gamma \frac{\partial s_{\mathrm{D}}}{\partial \rho}\bigg|_{\rho=1} = -1 + \frac{\gamma}{2\alpha}\frac{\mathrm{d}s_{\mathrm{wD}}}{\mathrm{d}\tau}. \tag{43}$$

The IBVP (39)–(43) can be solved straightforwardly after applying the Laplace transform with respect to τ, i.e., $\mathscr{L}\{(\circ), \tau \to p\} = \int_0^\infty (\circ)\mathrm{e}^{-p\tau}\,\mathrm{d}\tau$. Laplace-transformed (39)–(41) implies that the solution in the Laplace domain, denoted by a bar, is of the form $\bar{s}_{\mathrm{D}} = AK_0(\sqrt{p})$, where $K_0(x)$ is the zero-order modified Bessel function of the second kind. The factor A follows from the Laplace-transformed (42) and (43).

The solutions in the Laplace domain for wellbore drawdown, drawdown at an observation well located at the dimensionless distance ρ from the pumping well, and the wellface flowrate are, respectively,

$$\begin{aligned}\bar{s}_{\mathrm{wD}}(p;\gamma,\alpha,\eta) &= \frac{1}{\gamma}\left\{p\left[\frac{\sqrt{p}K_1(\sqrt{p})}{K_0(\sqrt{p}) + \eta\sqrt{p}K_1(\sqrt{p})} + \frac{p}{2\alpha}\right]\right\}^{-1} \\ &= \frac{1}{\gamma}\bar{s}_{\mathrm{wD}}|_{\gamma=1},\end{aligned} \tag{44}$$

$$\begin{aligned}\bar{s}_{\mathrm{D}}(\rho,p;\gamma,\alpha,\eta) &= \bar{s}_{\mathrm{wD}}(p;\gamma,\alpha,\eta)\frac{K_0(\rho\sqrt{p})}{K_0(\sqrt{p}) + \eta\sqrt{p}K_1(\sqrt{p})} \\ &= \frac{1}{\gamma}\bar{s}_{\mathrm{D}}|_{\gamma=1}\end{aligned} \tag{45}$$

and

$$\begin{aligned}\bar{q}_{\mathrm{D}}(p;\alpha,\eta) &= -\gamma\bar{s}_{\mathrm{wD}}\left[\eta + \frac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})}\right]^{-1} \\ &= -\left\{p\left[\frac{\sqrt{p}K_1(\sqrt{p})}{K_0(\sqrt{p}) + \eta\sqrt{p}K_1(\sqrt{p})} + \frac{p}{2\alpha}\right]\right. \\ &\qquad \left.\cdot\left[\eta + \frac{K_0(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})}\right]\right\}^{-1},\end{aligned} \tag{46}$$

where $K_1(x)$ is the first-order modified Bessel function of the second kind.

The notation after the second equality sign in (44) simply emphasizes the structure of the solution – it involves the fraction $1/\gamma$ and a part $(\bar{s}_{\mathrm{wD}}|_{\gamma=1})$ that does not depend on γ. The solution (45) has a similar structure, whereas the solution (46) does not depend on γ.

The reader should note that, for $\eta = 0$, (44) and (45) reduce to the Papadopulos and Cooper [61] model; whereas for $\alpha = \eta = 0$ and large times (small-p limit, $\lim_{p\to 0}\sqrt{p}K_1(\sqrt{p}) = 1$), they reduce to the Theis [71] model.

For the remainder of the paper, it is assumed that the considered pumping test model has been identified as appropriate and thus no modeling errors are present – measurement errors are the only errors. Such identification may be based on the hydrologist's experience, on a model discrimination methodology [42], or on a formal model-selection criterion [7,46].

6. Measurement domains

6.1. Drawdown measurements

Consider drawdown measurements $s(r,t)$ in an observation well with a pressure transducer characterized by the maximum absolute measurement error $\Delta s_{\max}$ and the maximum allowable drawdown $s_{\max}$. The useful measurements with this instrument are those that fulfill

$$\Delta s_{\max} \leqslant s \leqslant s_{\max}. \tag{47}$$

Now, let us introduce two additional dimensionless numbers

$$\epsilon_s = \frac{\Delta s_{\max}}{r_{\mathrm{w}}} \tag{48}$$

and

$$\xi_s = \frac{s_{\max}}{r_{\mathrm{w}}}. \tag{49}$$

It follows from (47)–(49), (32), and (45) that

$$\gamma\epsilon_s \leqslant s_{\mathrm{D}}|_{\gamma=1} \leqslant \gamma\xi_s. \tag{50}$$

For drawdown measurements $s_{\mathrm{w}}(t)$ in the pumping well an analogous inequality arises

$$\gamma\epsilon_s \leqslant s_{\mathrm{wD}}|_{\gamma=1} \leqslant \gamma\xi_s. \tag{51}$$

6.2. Flowrate measurements

Consider flowrate measurements $q(t)$ in the pumping well with a flowmeter characterized by the maximum absolute measurement error $\Delta q_{\max}$ and the maximum allowable flow rate $q_{\max}$. With additional dimensionless numbers

$$\epsilon_q = \frac{\Delta q_{\max}}{Q} \tag{52}$$

and

$$\xi_q = \frac{q_{\max}}{Q} \tag{53}$$

we obtain the flowrate analog of (50) and (51)

$$\epsilon_q \leqslant q_{\mathrm{D}} \leqslant \xi_q. \tag{54}$$

7. Sensitivity analysis of the pumping test data

It follows from (32), (33) and (36), and (44)–(46) that

$$s_{\mathrm{w}}(t;T,S,\eta) = r_{\mathrm{w}} s_{\mathrm{wD}}(\tau(t;T,S),\gamma(T),\alpha(S),\eta), \tag{55}$$

$$s(r,t;T,S,\eta) = r_{\mathrm{w}} s_{\mathrm{D}}(\rho,\tau(t;T,S),\gamma(T),\alpha(S),\eta), \tag{56}$$

$$q(t;T,S,\eta) = Q q_{\mathrm{D}}(\tau(t;T,S),\alpha(S),\eta). \tag{57}$$

Eqs. (35), (37), and (38) imply that

$$\frac{\partial\tau}{\partial T} = \frac{\tau}{T}, \tag{58}$$

$$\frac{\partial\tau}{\partial S} = -\frac{\tau}{S}, \tag{59}$$

$$\frac{\partial\gamma}{\partial T} = \frac{\gamma}{T}, \tag{60}$$

$$\frac{\partial\alpha}{\partial S} = \frac{\alpha}{S}. \tag{61}$$

The logarithmic sensitivity, (4), of drawdown with respect to transmissivity follows via the chain rule from (32), (45), (56), (58), and (60)

$$\begin{aligned}\frac{T}{s}\frac{\partial s}{\partial T} &= \frac{T}{s_{\mathrm{D}}}\left[\frac{\partial s_{\mathrm{D}}}{\partial\tau}\frac{\partial\tau}{\partial T} + \frac{\partial s_{\mathrm{D}}}{\partial\gamma}\frac{\partial\gamma}{\partial T}\right]\\ &= \frac{\tau}{s_{\mathrm{D}}|_{\gamma=1}}\frac{\partial s_{\mathrm{D}}|_{\gamma=1}}{\partial\tau} + \frac{\gamma}{s_{\mathrm{D}}|_{\gamma=1}/\gamma}\frac{\partial}{\partial\gamma}\left(\frac{1}{\gamma}s_{\mathrm{D}}|_{\gamma=1}\right)\\ &= \frac{\tau}{s_{\mathrm{D}}|_{\gamma=1}}\frac{\partial s_{\mathrm{D}}|_{\gamma=1}}{\partial\tau} - 1.\end{aligned} \tag{62}$$

Similarly, chain rule and (32), (56), (59), and (61) imply that

$$\frac{S}{s}\frac{\partial s}{\partial S} = -\frac{\tau}{s_{\mathrm{D}}|_{\gamma=1}}\frac{\partial s_{\mathrm{D}}|_{\gamma=1}}{\partial\tau} + \frac{\alpha}{s_{\mathrm{D}}|_{\gamma=1}}\frac{\partial s_{\mathrm{D}}|_{\gamma=1}}{\partial\alpha}, \tag{63}$$

and (56) and (32) give

$$\frac{\eta}{s}\frac{\partial s}{\partial\eta} = \frac{\eta}{s_{\mathrm{D}}|_{\gamma=1}}\frac{\partial s_{\mathrm{D}}|_{\gamma=1}}{\partial\eta}. \tag{64}$$

Analogously, we obtain

$$\frac{T}{s_{\mathrm{w}}}\frac{\partial s_{\mathrm{w}}}{\partial T} = \frac{\tau}{s_{\mathrm{wD}}|_{\gamma=1}}\frac{\partial s_{\mathrm{wD}}|_{\gamma=1}}{\partial\tau} - 1, \tag{65}$$

$$\frac{S}{s_{\mathrm{w}}}\frac{\partial s_{\mathrm{w}}}{\partial S} = -\frac{\tau}{s_{\mathrm{wD}}|_{\gamma=1}}\frac{\partial s_{\mathrm{wD}}|_{\gamma=1}}{\partial\tau} + \frac{\alpha}{s_{\mathrm{wD}}|_{\gamma=1}}\frac{\partial s_{\mathrm{wD}}|_{\gamma=1}}{\partial\alpha}, \tag{66}$$

$$\frac{\eta}{s_{\mathrm{w}}}\frac{\partial s_{\mathrm{w}}}{\partial\eta} = \frac{\eta}{s_{\mathrm{wD}}|_{\gamma=1}}\frac{\partial s_{\mathrm{wD}}|_{\gamma=1}}{\partial\eta}, \tag{67}$$

$$\frac{T}{q}\frac{\partial q}{\partial T} = \frac{\tau}{q_{\mathrm{D}}}\frac{\partial q_{\mathrm{D}}}{\partial\tau}, \tag{68}$$

$$\frac{S}{q}\frac{\partial q}{\partial S} = -\frac{\tau}{q_{\mathrm{D}}}\frac{\partial q_{\mathrm{D}}}{\partial\tau} + \frac{\alpha}{q_{\mathrm{D}}}\frac{\partial q_{\mathrm{D}}}{\partial\alpha} \tag{69}$$

and

$$\frac{\eta}{q}\frac{\partial q}{\partial\eta} = \frac{\eta}{q_{\mathrm{D}}}\frac{\partial q_{\mathrm{D}}}{\partial\eta}. \tag{70}$$

Note that the logarithmic sensitivities of the drawdown in the pumping well, drawdown in an observation well, and the wellface flowrate to transmissivity, T, and storativity, S, specified in (62)–(70), are independent of γ

and, are thus independent of T, Q, and r_w. They depend on the wellbore storage factor, α, and the skin factor, η, and are only functions of dimensionless time, τ, and, in case of the observation well drawdown sensitivities, of the dimensionless distance to the pumping well, ρ. Furthermore, for the no-wellbore storage and no-skin case, i.e., for the Theis [71] and Hantush and Jacob [30] models, these logarithmic sensitivities are independent not only of T, Q, and r_w, but also of S and thus, in contrast to the corresponding traditional and normalized sensitivities ([54] and others), can be represented by a single type curve or a family of type curves indexed by ρ.

The normalized sensitivities (62)–(70) can be calculated by numerical inversion of the Laplace transform via the De Hoog algorithm [20] in the *Mathematica* environment [77]. In particular, all α and η derivatives in (62)–(70) are calculated by first taking these derivatives in the Laplace p-domain, and then inverting them numerically to the dimensionless time τ-domain. There is no need to explicitly list any α or η derivatives of (44), (45), or (46) – they can be calculated straightforwardly, or one may let *Mathematica* calculate them symbolically, as we do. Due to the zero initial conditions, the τ derivatives are calculated by numerically inverting to the τ-domain the appropriate products of p and (44), (45), or (46). For example,

$$\frac{\partial s_{wD}}{\partial \tau} = \mathscr{L}^{-1}\{p\bar{s}_{wD}(p; T, S, \eta),\ \ p \to \tau\}. \tag{71}$$

8. Pumping test plausible relative errors under the fixed absolute measurement error model

Although the plausible relative errors in aquifer parameter estimates for the simple measurement model of (25) can be calculated from (26), we obtain them directly from (23) utilizing the already calculated logarithmic sensitivities. Thus it follows from these equations and from (62) that

$$\begin{aligned}\frac{dT}{T} &= \frac{\Delta s_{max}}{s} \bigg/ \left(\frac{T}{s}\frac{\partial s}{\partial T}\right) \\ &= \frac{\Delta s_{max}}{r_w s_D|_{\gamma=1}/\gamma} \bigg/ \left(\frac{\tau}{s_D|_{\gamma=1}}\frac{\partial s_D|_{\gamma=1}}{\partial \tau} - 1\right)\end{aligned}$$

or

$$\frac{dT}{T} = \gamma\epsilon_s\left(\tau\frac{\partial s_D|_{\gamma=1}}{\partial \tau} - s_D|_{\gamma=1}\right)^{-1} = \gamma\epsilon_s\left(\frac{dT}{T}\bigg|_{\gamma\epsilon_s=1}\right). \tag{72}$$

Similarly, (23) and (25), and (63) and (64) lead to

$$\frac{dS}{S} = \gamma\epsilon_s\left(-\tau\frac{\partial s_D|_{\gamma=1}}{\partial \tau} + \alpha\frac{\partial s_D|_{\gamma=1}}{\partial \alpha}\right)^{-1} = \gamma\epsilon_s\left(\frac{dS}{S}\bigg|_{\gamma\epsilon_s=1}\right), \tag{73}$$

$$\frac{d\eta}{\eta} = \gamma\epsilon_s\left(\eta\frac{\partial s_D|_{\gamma=1}}{\partial \eta}\right)^{-1} = \gamma\epsilon_s\left(\frac{d\eta}{\eta}\bigg|_{\gamma\epsilon_s=1}\right). \tag{74}$$

The plausible relative errors in parameter estimates from wellbore drawdown measurements are analogous in form to the above-listed plausible relative errors in parameter estimates from drawdown measurements in an observation well, i.e.,

$$\frac{dT}{T} = \gamma\epsilon_s\left(\tau\frac{\partial s_{wD}|_{\gamma=1}}{\partial \tau} - s_{wD}|_{\gamma=1}\right)^{-1} = \gamma\epsilon_s\left(\frac{dT}{T}\bigg|_{\gamma\epsilon_s=1}\right), \tag{75}$$

$$\frac{dS}{S} = \gamma\epsilon_s\left(-\tau\frac{\partial s_{wD}|_{\gamma=1}}{\partial \tau} + \alpha\frac{\partial s_{wD}|_{\gamma=1}}{\partial \alpha}\right)^{-1} = \gamma\epsilon_s\left(\frac{dS}{S}\bigg|_{\gamma\epsilon_s=1}\right), \tag{76}$$

$$\frac{d\eta}{\eta} = \gamma\epsilon_s\left(\eta\frac{\partial s_{wD}|_{\gamma=1}}{\partial \eta}\right)^{-1} = \gamma\epsilon_s\left(\frac{d\eta}{\eta}\bigg|_{\gamma\epsilon_s=1}\right), \tag{77}$$

and so are the plausible relative errors in parameter estimates from the wellface flowrate measurements

$$\frac{dT}{T} = \epsilon_q\left(\tau\frac{\partial q_D}{\partial \tau}\right)^{-1} = \epsilon_q\frac{dT}{T}\bigg|_{\epsilon_q=1}, \tag{78}$$

$$\frac{dS}{S} = \epsilon_q\left(-\tau\frac{\partial q_D}{\partial \tau} + \alpha\frac{\partial q_D}{\partial \alpha}\right)^{-1} = \epsilon_q\frac{dS}{S}\bigg|_{\epsilon_q=1} \tag{79}$$

and

$$\frac{d\eta}{\eta} = \epsilon_q\left(\eta\frac{\partial q_D}{\partial \eta}\right)^{-1} = \epsilon_q\frac{d\eta}{\eta}\bigg|_{\epsilon_q=1}. \tag{80}$$

Note that one way to cut the plausible relative errors in transmissivity, storativity, and the skin factor estimated from the wellbore drawdown is to decrease the value of γ in (72)–(77). As follows from (38), to cut these plausible relative errors in half, one needs to re-run the pumping test with doubled pumping rate (which may not always be possible). Also note from (78)–(80) that doubling pumping rate will not effect more accurate parameter estimates from the wellface flowrate. Another way to cut the plausible relative errors, applicable to estimates from the wellbore drawdown as well as from wellface flowrate, is to cut the measurement errors (and thus ϵ_s or ϵ_q) by using more accurate measurement instruments. Cutting the maximum measurement errors (Δs_{max} or Δq_{max}) in half cuts in half the corresponding plausible relative errors.

9. Results

Although logarithmic sensitivities define the logarithmic correlations between parameters, (22), and plausible relative errors, (23), it is still instructive to plot all of them rather than just the sensitivities.

In Figs. 1 and 2, we plot the logarithmic sensitivities of wellbore drawdown and wellface flowrate to transmissivity, storativity, and skin factor; the corresponding plausible relative errors in the three parameters; and the corresponding logarithmic correlations between these parameters. Fig. 1 presents these relations for a small

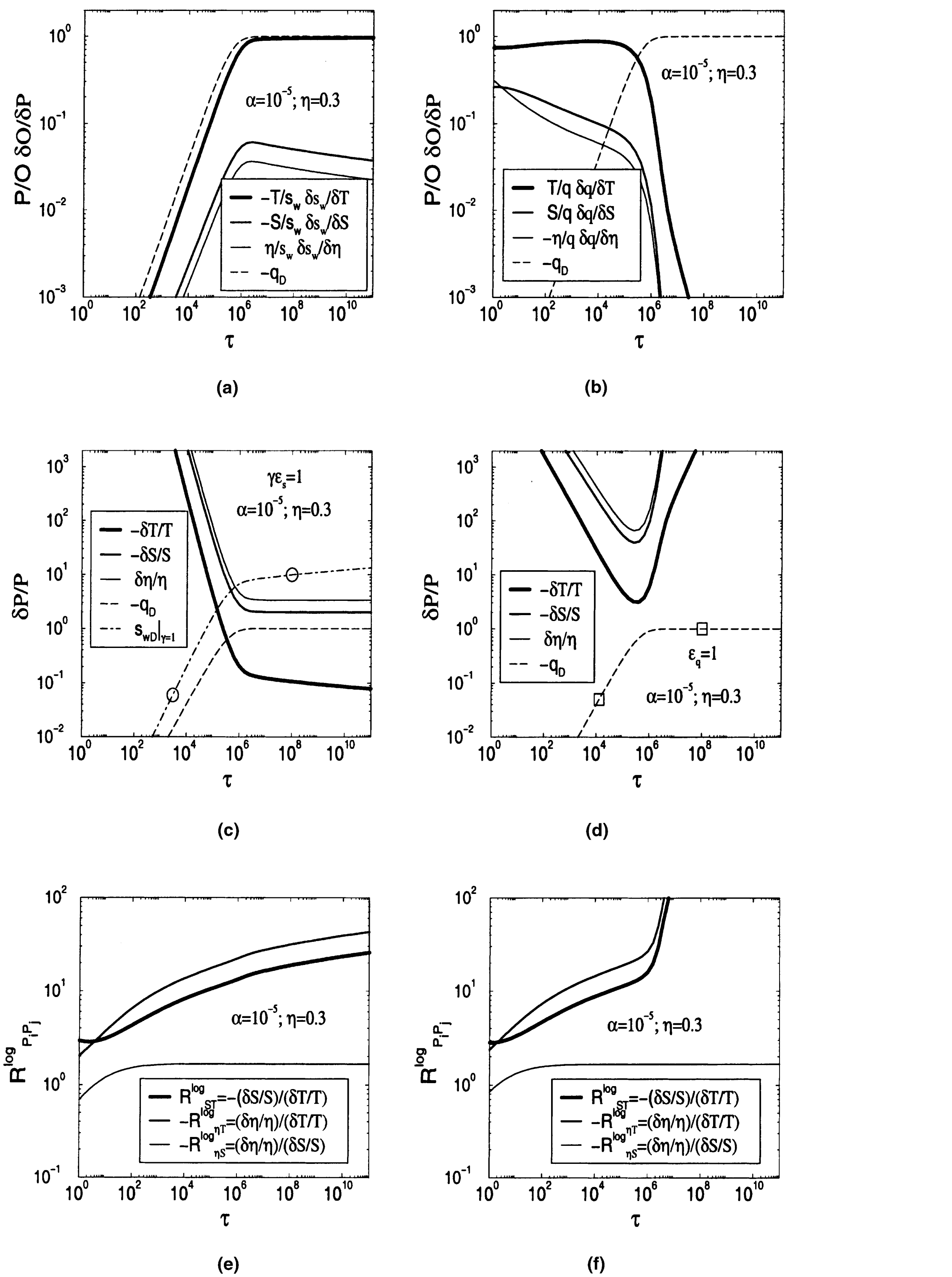

Fig. 1. (a) Sensitivities (with respect to transmissivity T, storativity S, and skin factor η) of wellbore drawdown, (b) sensitivities of wellface flowrate, (c) corresponding plausible relative errors from wellbore drawdown, (d) corresponding plausible relative errors from wellface flowrate, (e) logarithmic correlations from wellbore drawdown, and (f) logarithmic correlations from wellface flowrate; $\eta = 0.3$ and the wellbore storage $\alpha = Sr_w^2/r_c^2 = 10^{-5}$.

skin factor of $\eta = 0.3$, whereas Fig. 2 presents them for a large $\eta = 3$. For reference, to define the extent of the wellbore storage phase in the pumping test, we also plot there the dimensionless flowrate. It is apparent that the magnitude of the wellbore drawdown sensitivities grows rapidly with time during the wellbore storage phase

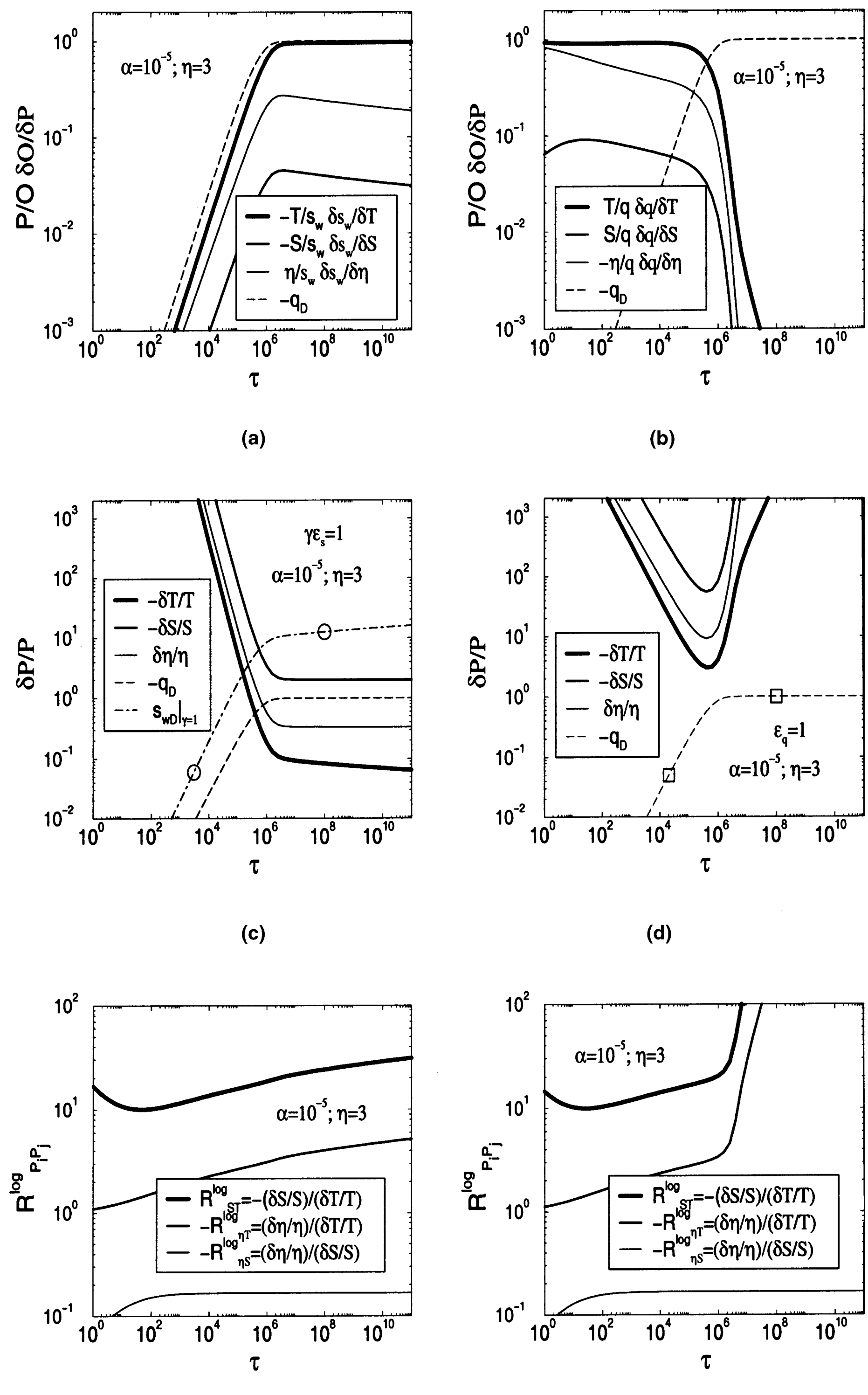

Fig. 2. (a) Sensitivities (with respect to transmissivity T, storativity S, and skin factor η) of wellbore drawdown, (b) sensitivities of wellface flowrate, (c) corresponding plausible relative errors from wellbore drawdown, (d) corresponding plausible relative errors from wellface flowrate, (e) logarithmic correlations from wellbore drawdown, and (f) logarithmic correlations from wellface flowrate; $\eta = 3$ and the wellbore storage $\alpha = Sr_w^2/r_c^2 = 10^{-5}$.

(until $-q_D \approx 1$) to reach a plateau or to begin a slow decrease. The wellface flowrate sensitivities behave differently. The magnitude of the T-sensitivity grows slowly throughout most of the wellbore storage phase, whereas the magnitudes of the S- and η-sensitivities keep mostly decreasing. As expected, the magnitudes of all three wellface flowrate sensitivities decay rapidly when the wellbore storage phase approaches an end.

For wellbore drawdown data and the small skin factor ($\eta \approx 0.3$), the magnitude of the S–T correlation, $R_{ST}^{\log}$, grows from about 3 to 30 throughout the considered time domain (Figs. 1(e)), whereas for the large skin factor ($\eta \approx 3$) it varies between 10 and 30 in the same time domain (Fig. 2(e)). For wellface flowrate data this correlation is similar during the wellbore storage phase (Figs. 1(f) and 2(f)); however, it rapidly increases afterwards. It follows from (22) that the magnitude of the relative error in S is larger than the magnitude of the relative error in T by the same factors.

For the small skin factor, the magnitude of the η–T correlation, $R_{\eta T}^{\log}$, behaves similarly to that of the S–T correlation for both the wellbore drawdown data and the wellface flowrate data from the wellbore storage phase (Figs. 1(e) and (f)). However, for the large skin factor, $R_{\eta T}^{\log}$ varies only between 1 and 5 throughout the considered time domain (Fig. 2(e)). It follows that more accurate estimates could be obtained for the large skin factors than for the small ones.

The η–S correlation, $R_{\eta S}^{\log} = -R_{\eta T}^{\log}/R_{ST}^{\log}$, follows from the other two correlations and is also plotted in Figs. 1(f) and 2(f).

It is evident from Figs. 1 and 2 that the plausible relative error in transmissivity is more than an order of magnitude smaller than the plausible relative error in storativity. For small skin factors ($\eta \approx 0.3$) the plausible relative error in the skin factor is larger than that in storativity, whereas for large skin factors ($\eta \approx 3$) it is smaller by more than half an order of magnitude.

The fallacy of the principle of maximum sensitivity for calibration curve model (25) is demonstrated in Figs. 1(b) and (d) and 2(b) and (d). It is apparent that flux measurements should not be made at very early times to estimate storativity and the skin factor as the principle of maximum sensitivity implies. Even though the flux sensitivities to storativity and skin factor are the largest at the early times (Figs. 1(b) and 2(b)), the flux measurements are biased by the very large relative errors at these times. It is clear from the corresponding plausible relative errors (Figs. 1(d) and 2(d)) that the flowrate measurements contain the most information about the parameters when the wellface flowrate reaches about half the total pumping rate ($-q_D \approx 0.5$ for $\eta = 0.3$ and $-q_D \approx 0.6$ for $\eta = 3$). Note that, at this stage, the corresponding plausible relative errors in parameter estimates from the wellbore drawdown (Figs. 1(c) and 2(c)) are still about half an order of magnitude larger than they become after the wellbore storage phase is over. Thus, the wellbore drawdown data should be collected only after the wellbore storage phase ends. Also, note that the principle of maximum sensitivity implies that taking drawdown measurements at any time after the wellbore storage phase is fine – the T-sensitivity is then constant and at its highest value (Fig. 1(a)). However, the plausible error in T keeps decreasing with time. In other words, contrary to the implications from the principle of maximum sensitivity, the longer one waits for the drawdown measurement the more information about transmissivity one obtains.

At times when appreciable drawdown can be recorded, as seen from Figs. 3 and 4, the relative magnitudes of T-, S-, and η-sensitivities for the drawdown in an observation well are analogous throughout most of

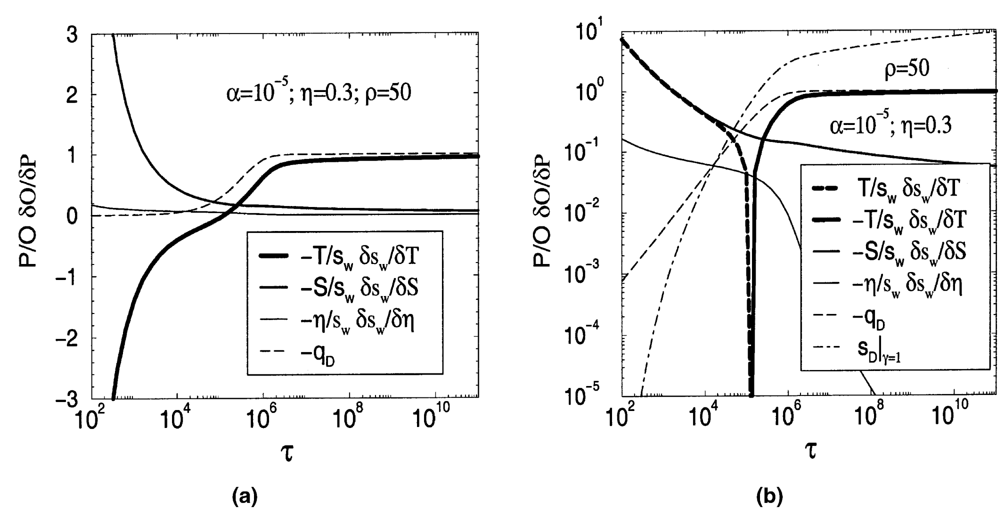

Fig. 3. Sensitivities (with respect to transmissivity T, storativity S, and skin factor η) of observation-well drawdown at the distance $\rho = 50$ from the pumping well for $\eta = 0.3$; (a) log–linear coordinates, (b) log–log coordinates.

the time domain to those for wellbore drawdown (Figs. 1 and 2). However, for an observation well, T-sensitivity undergoes a change of sign, whereas for wellbore drawdown it does not.

The variability of the T- and S-sensitivities with the wellbore storage factor is presented in Figs. 5 and 6, respectively. As expected, the smaller the wellbore storage parameter, α, the longer the wellbore storage phase for the sensitivities. The reader should note in Figs. 5(a) and 6(a) that the model sensitivities for $\alpha = \eta = 0$ do not reduce themselves fully to those of the Theis [71] model. They are only identical for sufficiently large times ($\tau > 5 \times 10^3$ for T-sensitivities and $\tau > 10^2$ for S-sensitivities). This is to be expected since the Theis [71] model is derived under a large time (small-p limit) simplifying assumption [31,36] that was not invoked in the derivation of the model (44)–(46) and its sensitivities.

Fig. 7 demonstrates that the magnitudes of T-sensitivities of wellbore drawdown decrease with increasing η throughout the wellbore storage phase and converge to the same value beyond this phase. The magnitudes of the T-sensitivities of wellface flowrate, on the other hand, grow with an increasing skin factor throughout most of the wellbore storage phase and decline rapidly at its end.

The magnitudes of both the S-sensitivities of wellbore drawdown and wellface flowrate decrease with increasing skin factor, as seen from Fig. 8.

10. Synthetic example 1

Consider a hypothetical pumping test with a constant total pumping rate of $Q = 5$ m^3/h $= 0.001389$ m^3/s con-

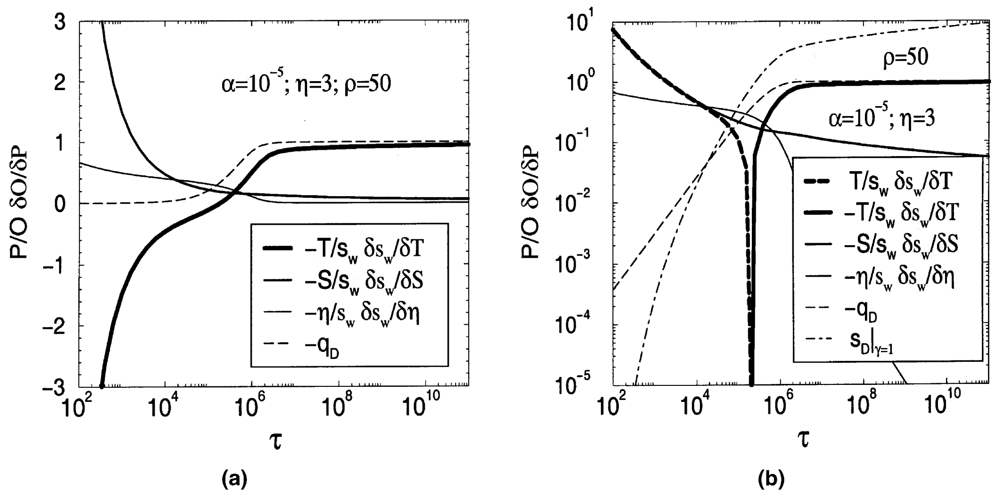

Fig. 4. Sensitivities (with respect to transmissivity T, storativity S, and skin factor η) of observation-well drawdown at the distance $\rho = 50$ from the pumping well for $\eta = 3$; (a) log–linear coordinates, (b) log–log coordinates.

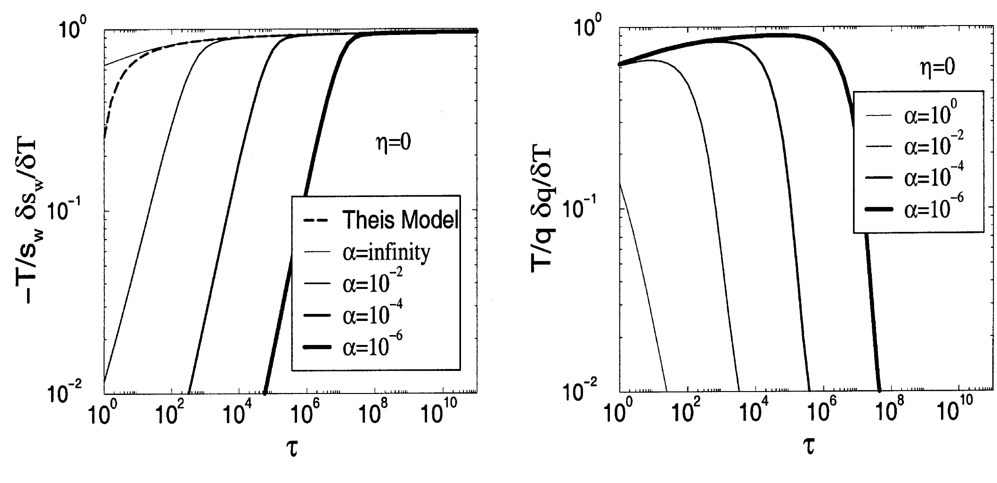

Fig. 5. Sensitivity with respect to transmissivity T (for a varying wellbore storage α) of (a) wellbore drawdown and (b) wellface flowrate.

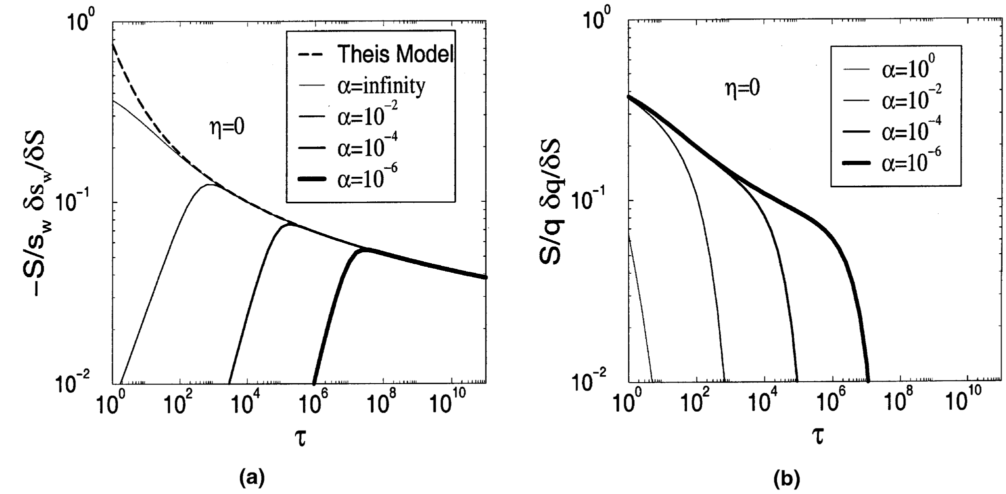

Fig. 6. Sensitivity with respect to storativity S (for a varying wellbore storage α) of (a) wellbore drawdown and (b) wellface flowrate.

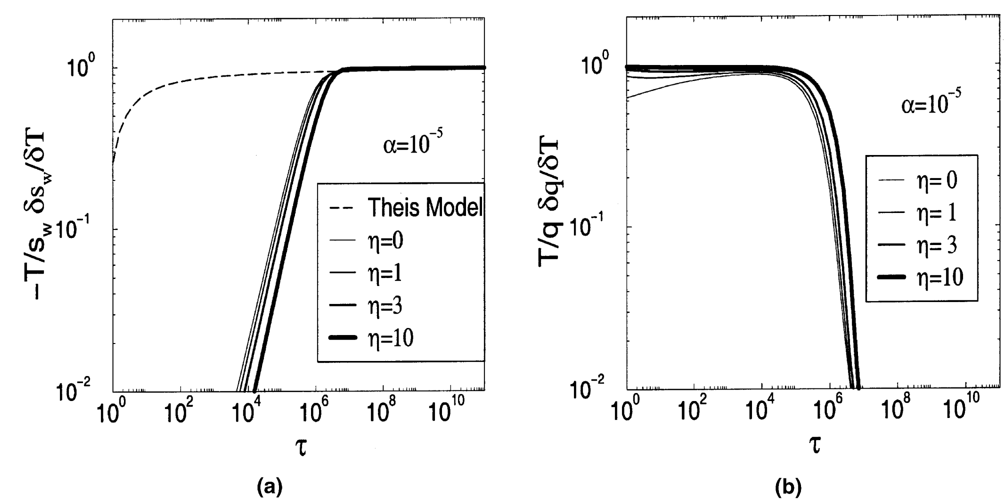

Fig. 7. Sensitivity with respect to transmissivity T (for a varying skin factor η) of (a) wellbore drawdown and (b) wellface flowrate.

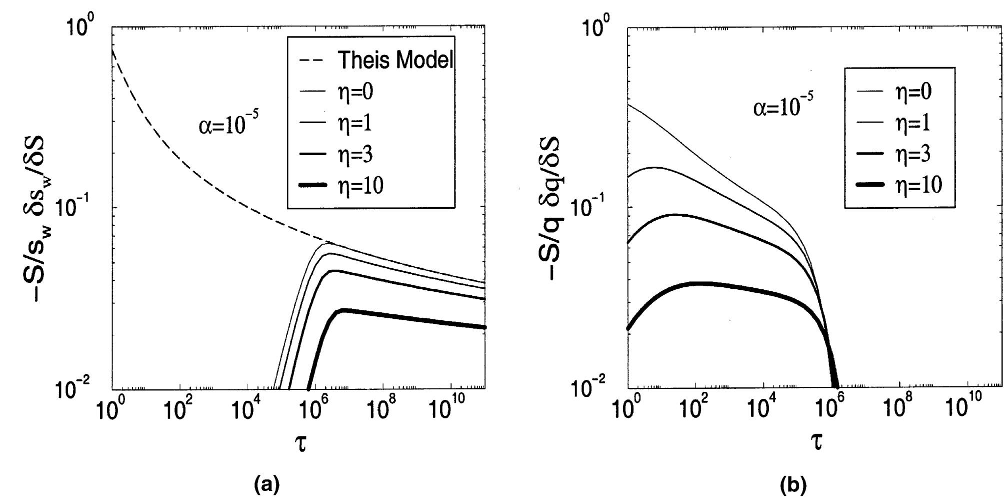

Fig. 8. Sensitivity with respect to storativity S (for a varying skin factor η) of (a) wellbore drawdown and (b) wellface flowrate.

ducted on a well of radius $r_w = r_c = 0.1$ m situated in a sandy confined aquifer. Assume that the test lasted $t_{max} = 6.4$ h and that the data were collected in the pumping well using

- a 30-psig pressure transducer with $\Delta s_{max} = 0.03$ m and $s_{max} = 21$ m, and
- a flowmeter with $\Delta q_{max} = 0.25$ m^3/h and $q_{max} = 20$ m^3/h.

Note that the flowmeter accuracy is identical to that of the state-of-the-art Tisco electromagnetic flowmeter.

Assume that the test can be described exactly by the pumping test model for a confined aquifer and that a type-curve or least-squares fitting of the collected wellbore drawdown data yields the following parameter estimates:

$$T = 4.37 \times 10^{-4}\ \mathrm{m}^2/\mathrm{s}, \quad \eta = 0.3, \quad \text{and } \alpha = S = 10^{-5}.$$

The related dimensionless numbers follow from (37), (38), (48), (49), (52), and (53):

$$\begin{aligned} &\alpha = 10^{-5}, \qquad \gamma = 0.1977, \qquad \epsilon_s = 0.3, \\ &\xi_s = 210, \qquad \epsilon_q = 0.05, \qquad \xi_q = 4. \end{aligned} \tag{81}$$

The dimensionless duration of the pumping test is about

$$\tau_{max} = \frac{Tt_{max}}{Sr_w^2} = 10^8. \tag{82}$$

In order to evaluate the plausible relative errors in the parameters, a set of type curves, such as those in Figs. 1 and 2, should be generated. However, since the estimated η and α in the example are identical to those in Fig. 1, we use the type curves contained in it.

As follows from (51) and (81), the meaningful wellbore drawdown measurements are those for times such that $\gamma\epsilon_s = 0.05931 \leqslant s_{wD}|_{\gamma=1} \leqslant \gamma\xi_s = 41.52$, which, in turn, along with Fig. 1(b) and (82), implies that $3.1 \times 10^3 \leqslant \tau \leqslant \tau_{max} = 10^8$. This time domain is marked by circles on the s_{wD} curve in Fig. 1(b).

Similarly, it follows from (54) and (81) that meaningful wellface flowrate measurements are those for times such that $0.05 \leqslant q_D \leqslant 4$, which, in turn, along with Fig. 1(d) and (82), implies that $1.3 \times 10^4 \leqslant \tau \leqslant \tau_{max} = 10^8$. This time domain is marked by squares on the q_D curve in Fig. 1(d).

As is apparent from Fig. 1(b), during most of the wellbore storage phase the plausible relative errors in parameter estimates from wellbore drawdown are huge, and thus, there is not much information about the parameters contained in the wellbore drawdown. Beyond the wellbore storage phase, the plausible relative errors in storativity and skin factor become constant with time, while the error in transmissivity continues to decrease slightly. From Fig. 1(b), plotted for $\gamma\epsilon_s = 1$, we read off the plausible relative errors in T, S, and η at time τ_{max}. They are 0.105, 1.97, and 3.30, respectively. As required by (75)–(77), we scale them by

$$\gamma\epsilon_s \approx 0.059$$

to obtain

$$\begin{aligned} &\frac{dT}{T} = 0.0062 \approx 0.6\%, \qquad \frac{dS}{S} = 0.116 \approx 12\%, \\ &\frac{d\eta}{\eta} = 0.195 = 19.5\%. \end{aligned} \tag{83}$$

As is apparent from Fig. 1(d), the plausible relative errors in parameter estimates from wellface flowrate reach a minimum in the middle of the wellbore storage phase ($-q_D \approx 0.5$), i.e., around $\tau = 2.7 \times 10^5$. If our parameters were estimated from the wellface flowrate collected around that time, then we can read off the plausible relative errors in T, S, and η from Fig. 1(d) (3.04, 38.7 and 64.9) and, as required by (78)–(80), scale them by $\epsilon_q \approx 0.05$ to obtain

$$\begin{aligned} &\frac{dT}{T} = 0.152 \approx 15\%, \qquad \frac{dS}{S} = 1.935 \approx 194\%, \\ &\frac{d\eta}{\eta} = 3.245 = 325\%. \end{aligned} \tag{84}$$

Note that the plausible relative errors in aquifer parameters from flowrate measurements are significantly larger than those from the drawdown measurements. This means that whenever both types of measurements are used, as is the case in the flowmeter test, the results will be biased by the dominant flowrate errors. The flowrate measurements here represent a bottleneck. This bottleneck is analogous to that in an IBVP solved numerically by a mixed scheme of first-order and second-order finite difference approximations – even when all governing equations and initial and boundary conditions are second-order accurate with the exception of just one equation or boundary condition that is first-order accurate, the solution remains first-order accurate rather than second-order accurate.

11. Synthetic example 2

Consider the same pumping test with the same instruments as in the previous example. Assume again that the test can be described exactly by the pumping test model for a confined aquifer; and this time, assume that a type-curve or least-squares fitting of the collected wellbore drawdown data (or wellface flowrate data) yields the following estimates:

$$T = 4.37 \times 10^{-4}\ \mathrm{m}^2/\mathrm{s}, \quad \eta = 3, \quad \text{and } \alpha = S = 10^{-5}.$$

The related dimensionless numbers are the same as in (81) and (82). Since the estimated η and α are identical to those used to generate Fig. 2, we use its type curves.

The meaningful wellbore drawdown measurements are again those for which $\gamma\epsilon_s = 0.05931 \leqslant s_{wD}|_{\gamma=1} \leqslant \gamma\xi_s = 41.52$, which, along with Fig. 2(b) and (82), implies that $2.8 \times 10^3 \leqslant \tau \leqslant \tau_{max} = 10^8$. Similarly, it follows from (54) and (81) that meaningful wellface flowrate measurements are those for times such that $0.05 \leqslant q_D \leqslant 4$, which, along with Fig. 2(d) and (82), implies that $2.0 \times 10^4 \leqslant \tau \leqslant \tau_{max} = 10^8$.

For the system parameters estimated from wellbore drawdown, we read off from Fig. 2(b), plotted for $\gamma\epsilon_s = 1$, the plausible relative errors in T, S, and η at time τ_{max} (0.078, 2.02, and 0.32). Again, as required by (75)–(77), we scale them by $\gamma\epsilon_s \approx 0.059$ to obtain

$$\frac{\mathrm{d}T}{T} = 0.0046 = 0.46\%, \qquad \frac{\mathrm{d}S}{S} = 0.12 = 12\%, \qquad \frac{\mathrm{d}\eta}{\eta} = 0.0189 = 1.9\%. \tag{85}$$

If our parameters were estimated from the wellface flowrate collected around the time when it reaches approximately half the total pumping rate ($\tau = 4.7 \times 10^5$), then we would read off the plausible relative errors in T, S, and η from Fig. 2(d) (2.87, 53.7, and 8.98) and, as required by (78)–(80), scale them by $\epsilon_q \approx 0.05$ to obtain

$$\frac{\mathrm{d}T}{T} = 0.144 = 14.4\%, \qquad \frac{\mathrm{d}S}{S} = 2.69 = 269\%, \qquad \frac{\mathrm{d}\eta}{\eta} = 0.449 = 44.9\%. \tag{86}$$

Again, the plausible relative errors in aquifer parameters from flowrate measurements are significantly larger than those from the drawdown measurement.

12. Extension to the model for a leaky aquifer

An extension to the pumping test in a leaky aquifer is straightforward [31]. Eq. (27) would be replaced by

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{S}{T}\frac{\partial s}{\partial t}, \tag{87}$$

where $B^2 = Kb/(K'/b')$, K' is the aquitard conductivity and b' is the aquitard thickness; one more dimensionless number would be introduced,

$$\beta = \frac{r_w}{B}, \tag{88}$$

all $\sqrt{p}$ in (44)–(46) would be replaced by $\sqrt{p + \beta^2}$, and expressions in (55)–(57) would depend on B or β. However, the relations (62)–(70) would remain unchanged.

13. Conclusions

The logarithmic sensitivity, (4), naturally arises in a normalized total differential. It can be interpreted as a transfer coefficient between the relative error in an input parameter and the relative error this input parameter alone induces in the output. All logarithmic sensitivities are dimensionless and thus can be compared to one another, as opposed to traditional sensitivities of an output to parameters of different dimensions and as opposed to normalized sensitivities of outputs of different dimensions.

Although the nature of the whole sensitivity matrix (Jacobian) **X**, (11), may need to be studied to determine the quality of the parameter estimates, the magnitudes of individual logarithmic sensitivities (4), deterministic parameter correlations (22), and plausible relative errors (23) may have, by themselves, significant implications for parameter estimation and sampling design. Combining the effects of logarithmic sensitivity and relative measurement error, plausible relative errors are especially useful when the relative measurement errors are not uniform through space and time, which is typically the case. Their consideration leads to:

> *Heuristic guidelines for measurement selection:*
>
> Minimizing the plausible relative errors rather than maximizing the corresponding sensitivities should serve as a guide to identifying the measurements most useful for parameter estimation or as candidate measurements for optimal sampling. Furthermore, avoiding among them as much as possible the measurements with the high parameter correlations may help ensure that the sensitivity matrix **X** (11) (or $\mathbf{X}^T\mathbf{X}$) is well-conditioned and thus, that the parameter estimates are accurate.

That one should minimize the plausible relative errors rather than maximize the sensitivities is clearly demonstrated in Figs. 1(b) and 2(b), which present the sensitivities of wellface flowrate with respect to storativity and skin factor as decreasing throughout most of the time domain. All of these sensitivities reach the near-maximum values for impractically small times, during which the pressure transducer or the flowmeter measurements are biased by the largest relative measurement errors. The corresponding plausible relative errors in parameter estimates (Figs. 1(d) and 2 (d)) clearly identify the later-time points around which the measurements contain the most information about the parameters.

A model, accounting for the wellbore storage and infinitesimal skin, of a pumping test conducted on a fully penetrating well situated in a confined aquifer is considered. Logarithmic sensitivities of its drawdown and wellface flowrate are calculated. The logarithmic sensitivities of drawdown in the pumping well, drawdown in an observation well, and wellface flowrate to transmissivity, T, storativity, S, and the skin factor, η, specified in (62)–(70), are independent of T, Q, and r_w.

They depend on the wellbore storage factor, α, and the skin factor, η, and are functions of dimensionless time, τ. The observation well drawdown sensitivities also depend on dimensionless distance to the pumping well, ρ. Furthermore, for the no-wellbore storage and no-skin case, i.e., for the Theis [71] and Hantush and Jacob [30] models, these logarithmic sensitivities are independent of S. Thus, in contrast to the corresponding traditional and normalized sensitivities [18,54,57], the logarithmic sensitivities can be represented by a single type curve or a family of type curves.

Particular conclusions from this research include:

1. The magnitude of the wellbore drawdown sensitivities grows rapidly with time during the wellbore storage phase (until the wellface flowrate, q, equals approximately the pumping rate, Q, i.e., until $-q_D = q/Q \approx 1$) and then reaches a plateau or begins a slow decrease. The magnitudes of the wellface flowrate sensitivities behave differently. T-sensitivity grows slowly throughout most of the wellbore storage phase, whereas S- and η-sensitivities mostly decrease. The magnitudes of all three wellface flowrate sensitivities decay rapidly when the wellbore storage phase approaches an end.
2. The magnitudes of T-sensitivities of wellbore drawdown decrease with increasing skin factor, η, throughout the wellbore storage phase and converge to a single value beyond this phase. The magnitudes of the T-sensitivities of wellface flowrate, on the other hand, grow with an increasing skin factor throughout most of the wellbore storage phase and decline rapidly at its end.
3. The magnitudes of the S-sensitivities of both wellbore drawdown and wellface flowrate decrease with an increasing skin factor.
4. The magnitude of the S–T correlation, $R_{ST}^{\log}$, for wellbore drawdown data mostly increases with time and at large times exceeds 10–30. For wellface flowrate data, $R_{ST}^{\log}$ is similar during the wellbore storage phase and rapidly increases afterwards. For the small skin factor, the magnitude of the η–T correlation, $R_{\eta T}^{\log}$, behaves similarly to that of the S–T correlation. For the large skin factor, $R_{\eta T}^{\log}$ varies only up to half an order of magnitude. Thus, more accurate estimates could be obtained for the large skin factors than for the small ones. The η–S correlation follows via (22) from the other two, i.e., $R_{\eta S}^{\log} = -R_{\eta T}^{\log}/R_{ST}^{\log}$.
5. At times when appreciable drawdown can be recorded, the relative magnitudes of T-, S-, and η-sensitivities for the drawdown in an observation well are analogous throughout most of the domain to those for wellbore drawdown. However, for an observation well, the T-sensitivity undergoes a change of sign, whereas for wellbore drawdown it does not.
6. The plausible relative errors in T, S, and η estimated from wellbore drawdown rapidly decrease during the wellbore storage phase (until $-q_D \approx 1$); the S- and η-errors reach a plateau, whereas the T-errors continue to slowly decrease outside the wellbore storage phase. On the other hand, the plausible relative errors in T, S, and η estimated from wellface flowrate rapidly decrease during the wellbore storage phase before reaching a minimum (when $-q_D \approx 0.5$) and then rapidly increase. The *practical implication* for the transient flowmeter test is that measurements of drawdown and wellface flowrate should not be made during the early times of the wellbore storage phase (when $-q_D \ll 0.5$). This conclusion is also expected to be true for the multi-stage layered reservoir tests used in petroleum engineering [67].
7. The magnitude of the plausible relative error in T estimated from wellbore drawdown and from wellface flowrate is about an order of magnitude smaller than the corresponding plausible relative error in S. The corresponding plausible relative error in η decreases with increasing skin factor. For small η (<0.3), it is even larger than the plausible relative error in S, whereas for large η (>3), it may approach the plausible relative error in T.
8. Other *practical implications* follow from (38), and (72)–(77) – the plausible relative errors in transmissivity, storativity, and skin factor estimated from wellbore drawdown can be cut by an arbitrary factor (within the limits of applicability of the model) through increasing the pumping rate by this factor (in the next test). However, it follows from (78)–(80) that doubling pumping rate will not effect more accurate parameter estimates from the wellface flowrate. The plausible relative errors in these parameters estimated from either wellbore drawdown or wellface flowrate can also be decreased by using more accurate drawdown or flowrate measurement instruments. Cutting their maximum measurement errors ($\Delta s_{\max}$ or $\Delta q_{\max}$) by a factor cuts the corresponding plausible relative errors by the same factor.
9. The plausible relative errors in aquifer parameters from flowrate measurements are significantly larger than those from the drawdown measurements. Thus, the flowrate measurements represent a bottleneck of the flowmeter test. In order to improve flowmeter test measurements, more accurate flowmeters need to be built.

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