Cvičení: Ritzova metoda

10. dubna 2020

Úloha. Řešte okrajovou úlohu

$$-u''(x) + (-4 + 2x)\, u(x) = -3x; \quad x \in \langle 0, 1 \rangle\, ; \quad u(0) = 0, \;\; u(1) = 0,$$

Ritzovou metodou s volbou báze

$$v_1(x) \;=\; x(1-x), \quad v_2(x) \;=\; x^2(1-x).$$

Řešení.

- první derivace bázových funkcí:

$$\begin{aligned} v_1(x) &= x - x^2, & v_1'(x) &= 1 - 2x, \\ v_2(x) &= x^2 - x^3, & v_2'(x) &= 2x - 3x^2. \end{aligned}$$

- energetický součin:

$$\begin{aligned} (u, v)_A &= \int_0^1 \left(-u''(x) + (-4+2x)\, u(x)\right) v(x)\, \mathrm{d}x \;=\; -\int_0^1 u''(x) v(x)\, \mathrm{d}x + \int_0^1 (-4+2x)\, u(x) v(x)\, \mathrm{d}x \\ &= \int_0^1 u'(x) v'(x)\, \mathrm{d}x + \int_0^1 (-4+2x)\, u(x) v(x)\, \mathrm{d}x \end{aligned}$$

- koeficienty matice soustavy rovnic:

$$\begin{aligned} (v_1, v_1)_A &= \int_0^1 (1-2x)\cdot(1-2x)\ \mathrm{d}x + \int_0^1 (-4+2x)\cdot\left(x-x^2\right)\cdot\left(x-x^2\right)\ \mathrm{d}x \\ &= \int_0^1 \left(1 - 4x + 4x^2\right)\ \mathrm{d}x + \int_0^1 \left(-4x^2 + 10x^3 - 8x^4 + 2x^5\right)\ \mathrm{d}x \\ &= \left[x - 2x^2 + \frac{4}{3}x^3\right]_0^1 + \left[-\frac{4}{3}x^3 + \frac{5}{2}x^4 - \frac{8}{5}x^5 + \frac{1}{3}x^6\right]_0^1 \\ &= \frac{1}{3} + -\frac{1}{10} \;=\; \frac{7}{30} \;\doteq\; 0.233 \end{aligned}$$

$$\begin{aligned} (v_1, v_2)_A &= \int_0^1 (1-2x)\cdot\left(2x-3x^2\right)\ \mathrm{d}x + \int_0^1 (-4+2x)\cdot\left(x-x^2\right)\cdot\left(x^2-x^3\right)\ \mathrm{d}x \\ &= \int_0^1 \left(2x - 7x^2 + 6x^3\right)\ \mathrm{d}x + \int_0^1 \left(-4x^3 + 10x^4 - 8x^5 + 2x^6\right)\ \mathrm{d}x \\ &= \left[x^2 - \frac{7}{3}x^3 + \frac{3}{2}x^4\right]_0^1 + \left[-x^4 + 2x^5 - \frac{4}{3}x^6 + \frac{2}{7}x^7\right]_0^1 \\ &= \frac{1}{6} + -\frac{1}{21} \;=\; \frac{5}{42} \;\doteq\; 0.119 \end{aligned}$$

$$
\begin{aligned}
(v_2, v_2)_A &= \int_0^1 \left(2x - 3x^2\right) \cdot \left(2x - 3x^2\right) \, \mathrm{d}x + \int_0^1 (-4 + 2x) \cdot \left(x^2 - x^3\right) \cdot \left(x^2 - x^3\right) \, \mathrm{d}x \\
&= \int_0^1 \left(4x^2 - 12x^3 + 9x^4\right) \, \mathrm{d}x + \int_0^1 \left(-4x^4 + 10x^5 - 8x^6 + 2x^7\right) \, \mathrm{d}x \\
&= \left[\frac{4}{3}x^3 - 3x^4 + \frac{9}{5}x^5\right]_0^1 + \left[-\frac{4}{5}x^5 + \frac{5}{3}x^6 - \frac{8}{7}x^7 + \frac{1}{4}x^8\right]_0^1 \\
&= \frac{2}{15} + -\frac{11}{420} \;=\; \frac{3}{28} \;\doteq\; 0.107
\end{aligned}
$$

- koeficienty pravé strany soustavy rovnic:

$$
(f, v_1) \;=\; \int_0^1 (-3x) \cdot \left(x - x^2\right) \, \mathrm{d}x \;=\; \int_0^1 \left(-3x^2 + 3x^3\right) \, \mathrm{d}x \;=\; \left[-x^3 + \frac{3}{4}x^4\right]_0^1 \;=\; -\frac{1}{4} \doteq -0.25
$$

$$
(f, v_2) \;=\; \int_0^1 (-3x) \cdot \left(x^2 - x^3\right) \, \mathrm{d}x \;=\; \int_0^1 \left(-3x^3 + 3x^4\right) \, \mathrm{d}x \;=\; \left[-\frac{3}{4}x^4 + \frac{3}{5}x^5\right]_0^1 \;=\; -\frac{3}{20} \doteq -0.15
$$

- rozšířená matice soustavy rovnic:

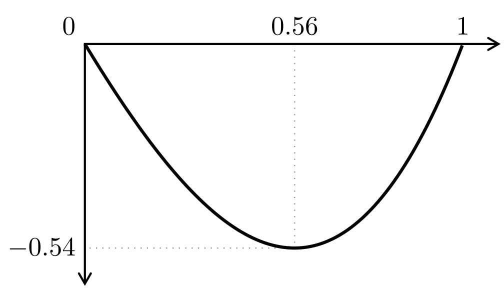
$$
\left(\begin{array}{cc|c} \frac{7}{30} & \frac{5}{42} & -\frac{1}{4} \\ \frac{5}{42} & \frac{3}{28} & -\frac{3}{20} \end{array}\right) \;\doteq\; \left(\begin{array}{cc|c} 0.233 & 0.119 & -0.25 \\ 0.119 & 0.107 & -0.15 \end{array}\right)
$$

- řešení soustavy rovnic:

$$
\alpha_1 \;=\; -\frac{315}{382} \;\doteq\; -0.825, \qquad \alpha_2 \;=\; -\frac{462}{955} \;\doteq\; -0.484
$$

- aproximace řešení:

$$
\tilde{u}(x) \;=\; -0.825 \cdot \left(x - x^2\right) - 0.484 \cdot \left(x^2 - x^3\right) \;=\; -1.649x + 0.682x^2 + 0.968x^3
$$



Úloha. Řešte okrajovou úlohu

$$
-u''(x) + (4 - x)\, u(x) = 2 - x^3; \quad x \in \langle 0, 2 \rangle\,; \quad u(0) = 0, \;\; u(2) = 0,
$$

Ritzovou metodou s volbou báze

$$
v_1(x) \;=\; x(2 - x), \quad v_2(x) \;=\; x^2(2 - x).
$$

Řešení.

- první derivace bázových funkcí:

$$
\begin{aligned}
v_1(x) &= 2x - x^2, & v_1'(x) &= 2 - 2x, \\
v_2(x) &= 2x^2 - x^3, & v_2'(x) &= 4x - 3x^2.
\end{aligned}
$$

- energetický součin:

$$\begin{aligned}(u,v)_A &= \int_0^2 \left(-u''(x) + (4-x)\,u(x)\right) v(x)\,\mathrm{d}x = -\int_0^2 u''(x)v(x)\,\mathrm{d}x + \int_0^2 (4-x)\,u(x)v(x)\,\mathrm{d}x \\ &= \int_0^2 u'(x)v'(x)\,\mathrm{d}x + \int_0^2 (4-x)\,u(x)v(x)\,\mathrm{d}x\end{aligned}$$

- koeficienty matice soustavy rovnic:

$$\begin{aligned}(v_1,v_1)_A &= \int_0^2 (2-2x)\cdot(2-2x)\,\mathrm{d}x + \int_0^2 (4-x)\cdot\left(2x-x^2\right)\cdot\left(2x-x^2\right)\,\mathrm{d}x \\ &= \int_0^2 \left(4-8x+4x^2\right)\mathrm{d}x + \int_0^2 \left(16x^2-20x^3+8x^4-x^5\right)\mathrm{d}x \\ &= \left[4x-4x^2+\frac{4}{3}x^3\right]_0^2 + \left[\frac{16}{3}x^3-5x^4+\frac{8}{5}x^5-\frac{1}{6}x^6\right]_0^2 \\ &= \frac{8}{3}+\frac{16}{5} = \frac{88}{15} \doteq 5.867\end{aligned}$$

$$\begin{aligned}(v_1,v_2)_A &= \int_0^2 (2-2x)\cdot\left(4x-3x^2\right)\mathrm{d}x + \int_0^2 (4-x)\cdot\left(2x-x^2\right)\cdot\left(2x^2-x^3\right)\mathrm{d}x \\ &= \int_0^2 \left(8x-14x^2+6x^3\right)\mathrm{d}x + \int_0^2 \left(16x^3-20x^4+8x^5-x^6\right)\mathrm{d}x \\ &= \left[4x^2-\frac{14}{3}x^3+\frac{3}{2}x^4\right]_0^2 + \left[4x^4-4x^5+\frac{4}{3}x^6-\frac{1}{7}x^7\right]_0^2 \\ &= \frac{8}{3}+\frac{64}{21} = \frac{40}{7} \doteq 5.714\end{aligned}$$

$$\begin{aligned}(v_2,v_2)_A &= \int_0^2 \left(4x-3x^2\right)\cdot\left(4x-3x^2\right)\mathrm{d}x + \int_0^2 (4-x)\cdot\left(2x^2-x^3\right)\cdot\left(2x^2-x^3\right)\mathrm{d}x \\ &= \int_0^2 \left(16x^2-24x^3+9x^4\right)\mathrm{d}x + \int_0^2 \left(16x^4-20x^5+8x^6-x^7\right)\mathrm{d}x \\ &= \left[\frac{16}{3}x^3-6x^4+\frac{9}{5}x^5\right]_0^2 + \left[\frac{16}{5}x^5-\frac{10}{3}x^6+\frac{8}{7}x^7-\frac{1}{8}x^8\right]_0^2 \\ &= \frac{64}{15}+\frac{352}{105} = \frac{160}{21} \doteq 7.619\end{aligned}$$

- koeficienty pravé strany soustavy rovnic:

$$(f,v_1) = \int_0^2 \left(2-x^3\right)\cdot\left(2x-x^2\right)\mathrm{d}x = \int_0^2 \left(4x-2x^2-2x^4+x^5\right)\mathrm{d}x = \left[2x^2-\frac{2}{3}x^3-\frac{2}{5}x^5+\frac{1}{6}x^6\right]_0^2 = \frac{8}{15} \doteq 0.5$$

$$(f,v_2) = \int_0^2 \left(2-x^3\right)\cdot\left(2x^2-x^3\right)\mathrm{d}x = \int_0^2 \left(4x^2-2x^3-2x^5+x^6\right)\mathrm{d}x = \left[\frac{4}{3}x^3-\frac{1}{2}x^4-\frac{1}{3}x^6+\frac{1}{7}x^7\right]_0^2 = -\frac{8}{21} \doteq$$

- rozšířená matice soustavy rovnic:

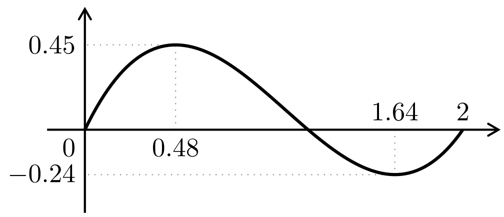
$$\left(\begin{array}{cc|c} \frac{88}{15} & \frac{40}{7} & \frac{8}{15} \\ \frac{40}{7} & \frac{160}{21} & -\frac{8}{21}\end{array}\right) \doteq \left(\begin{array}{cc|c} 5.867 & 5.714 & 0.533 \\ 5.714 & 7.619 & -0.381\end{array}\right)$$

- řešení soustavy rovnic:

$$\alpha_1 = \frac{43}{83} \doteq 0.518, \qquad \alpha_2 = -\frac{182}{415} \doteq -0.439$$

- aproximace řešení:

$$\tilde{u}(x) = 0.518\cdot\left(2x-x^2\right) - 0.439\cdot\left(2x^2-x^3\right) = 2.072x - 2.79x^2 + 0.877x^3$$



Úloha. Řešte okrajovou úlohu

$$-u''(x) + (-2+2x)\,u(x) = -2x^2 + x^3; \quad x \in \langle 0, 3\rangle\,; \quad u(0) = 0, \;\; u(3) = 0,$$

Ritzovou metodou s volbou báze

$$v_1(x) \;=\; x(3-x), \quad v_2(x) \;=\; x^2(3-x).$$

Řešení.

- první derivace bázových funkcí:

$$\begin{aligned} v_1(x) &= 3x - x^2, & v_1'(x) &= 3 - 2x, \\ v_2(x) &= 3x^2 - x^3, & v_2'(x) &= 6x - 3x^2. \end{aligned}$$

- energetický součin:

$$\begin{aligned} (u,v)_A &= \int_0^3 \left(-u''(x) + (-2+2x)\,u(x)\right) v(x)\,\mathrm{d}x \;=\; -\int_0^3 u''(x)v(x)\,\mathrm{d}x + \int_0^3 (-2+2x)\,u(x)v(x)\,\mathrm{d}x \\ &= \int_0^3 u'(x)v'(x)\,\mathrm{d}x + \int_0^3 (-2+2x)\,u(x)v(x)\,\mathrm{d}x \end{aligned}$$

- koeficienty matice soustavy rovnic:

$$\begin{aligned} (v_1,v_1)_A &= \int_0^3 (3-2x)\cdot(3-2x)\;\mathrm{d}x + \int_0^3 (-2+2x)\cdot\left(3x-x^2\right)\cdot\left(3x-x^2\right)\;\mathrm{d}x \\ &= \int_0^3 \left(9 - 12x + 4x^2\right)\;\mathrm{d}x + \int_0^3 \left(-18x^2 + 30x^3 - 14x^4 + 2x^5\right)\;\mathrm{d}x \\ &= \left[9x - 6x^2 + \frac{4}{3}x^3\right]_0^3 + \left[-6x^3 + \frac{15}{2}x^4 - \frac{14}{5}x^5 + \frac{1}{3}x^6\right]_0^3 \\ &= 9 + \frac{81}{10} \;=\; \frac{171}{10} \;\doteq\; 17.1 \end{aligned}$$

$$\begin{aligned} (v_1,v_2)_A &= \int_0^3 (3-2x)\cdot\left(6x-3x^2\right)\;\mathrm{d}x + \int_0^3 (-2+2x)\cdot\left(3x-x^2\right)\cdot\left(3x^2-x^3\right)\;\mathrm{d}x \\ &= \int_0^3 \left(18x - 21x^2 + 6x^3\right)\;\mathrm{d}x + \int_0^3 \left(-18x^3 + 30x^4 - 14x^5 + 2x^6\right)\;\mathrm{d}x \\ &= \left[9x^2 - 7x^3 + \frac{3}{2}x^4\right]_0^3 + \left[-\frac{9}{2}x^4 + 6x^5 - \frac{7}{3}x^6 + \frac{2}{7}x^7\right]_0^3 \\ &= \frac{27}{2} + \frac{243}{14} \;=\; \frac{216}{7} \;\doteq\; 30.857 \end{aligned}$$

$$\begin{aligned} (v_2,v_2)_A &= \int_0^3 \left(6x-3x^2\right)\cdot\left(6x-3x^2\right)\;\mathrm{d}x + \int_0^3 (-2+2x)\cdot\left(3x^2-x^3\right)\cdot\left(3x^2-x^3\right)\;\mathrm{d}x \\ &= \int_0^3 \left(36x^2 - 36x^3 + 9x^4\right)\;\mathrm{d}x + \int_0^3 \left(-18x^4 + 30x^5 - 14x^6 + 2x^7\right)\;\mathrm{d}x \\ &= \left[12x^3 - 9x^4 + \frac{9}{5}x^5\right]_0^3 + \left[-\frac{18}{5}x^5 + 5x^6 - 2x^7 + \frac{1}{4}x^8\right]_0^3 \\ &= \frac{162}{5} + \frac{729}{20} \;=\; \frac{1377}{20} \;\doteq\; 68.85 \end{aligned}$$

- koeficienty pravé strany soustavy rovnic:

$$(f, v_1) \;=\; \int_0^3 \left(-2x^2 + x^3\right) \cdot \left(3x - x^2\right)\,\mathrm{d}x \;=\; \int_0^3 \left(-6x^3 + 5x^4 - x^5\right)\,\mathrm{d}x \;=\; \left[-\frac{3}{2}x^4 + x^5 - \frac{1}{6}x^6\right]_0^3 \;=\; 0 \doteq 0.0$$

$$(f, v_2) \;=\; \int_0^3 \left(-2x^2 + x^3\right) \cdot \left(3x^2 - x^3\right)\,\mathrm{d}x \;=\; \int_0^3 \left(-6x^4 + 5x^5 - x^6\right)\,\mathrm{d}x \;=\; \left[-\frac{6}{5}x^5 + \frac{5}{6}x^6 - \frac{1}{7}x^7\right]_0^3 \;=\; \frac{243}{70} \doteq 3.471$$

- rozšířená matice soustavy rovnic:

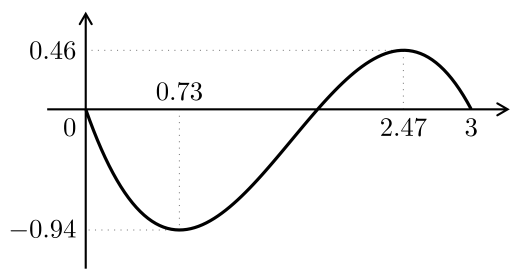
$$\left(\begin{array}{cc|c} \frac{171}{10} & \frac{216}{7} & 0 \\ \frac{216}{7} & \frac{1377}{20} & \frac{243}{70} \end{array}\right) \;\doteq\; \left(\begin{array}{cc|c} 17.1 & 30.857 & 0.0 \\ 30.857 & 68.85 & 3.471 \end{array}\right)$$

- řešení soustavy rovnic:

$$\alpha_1 \;=\; -\frac{480}{1009} \;\doteq\; -0.476, \qquad \alpha_2 \;=\; \frac{266}{1009} \;\doteq\; 0.264$$

- aproximace řešení:

$$\tilde{u}(x) \;=\; -0.476 \cdot \left(3x - x^2\right) + 0.264 \cdot \left(3x^2 - x^3\right) \;=\; -2.854x + 2.533x^2 - 0.527x^3$$



Úloha. Řešte okrajovou úlohu

$$-u''(x) + \left(2x + 2x^2\right) u(x) = -3; \quad x \in \langle 0, 2 \rangle\,; \quad u(0) = 0, \;\; u(2) = 0,$$

Ritzovou metodou s volbou báze

$$v_1(x) \;=\; x(2 - x), \quad v_2(x) \;=\; x^2(2 - x), \quad v_3(x) \;=\; x^3(2 - x).$$

Řešení.

- první derivace bázových funkcí:

$$\begin{aligned} v_1(x) &= 2x - x^2, & v_1'(x) &= 2 - 2x, \\ v_2(x) &= 2x^2 - x^3, & v_2'(x) &= 4x - 3x^2, \\ v_3(x) &= 2x^3 - x^4, & v_3'(x) &= 6x^2 - 4x^3. \end{aligned}$$

- energetický součin:

$$\begin{aligned} (u, v)_A &= \int_0^2 \left(-u''(x) + \left(2x + 2x^2\right) u(x)\right) v(x)\,\mathrm{d}x \\ &= -\int_0^2 u''(x) v(x)\,\mathrm{d}x + \int_0^2 \left(2x + 2x^2\right) u(x) v(x)\,\mathrm{d}x \\ &= \int_0^2 u'(x) v'(x)\,\mathrm{d}x + \int_0^2 \left(2x + 2x^2\right) u(x) v(x)\,\mathrm{d}x \end{aligned}$$

- koeficienty matice soustavy rovnic:

$$\begin{aligned}
(v_1, v_1)_A &= \int_0^2 (2-2x)\cdot(2-2x)\ \mathrm{d}x + \int_0^2 \left(2x+2x^2\right)\cdot\left(2x-x^2\right)\cdot\left(2x-x^2\right)\ \mathrm{d}x \\
&= \int_0^2 \left(4-8x+4x^2\right)\ \mathrm{d}x + \int_0^2 \left(8x^3-6x^5+2x^6\right)\ \mathrm{d}x \\
&= \left[4x-4x^2+\frac{4}{3}x^3\right]_0^2 + \left[2x^4-x^6+\frac{2}{7}x^7\right]_0^2 \\
&= \frac{8}{3}+\frac{32}{7} \ = \ \frac{152}{21} \ \doteq \ 7.238
\end{aligned}$$

$$\begin{aligned}
(v_1, v_2)_A &= \int_0^2 (2-2x)\cdot\left(4x-3x^2\right)\ \mathrm{d}x + \int_0^2 \left(2x+2x^2\right)\cdot\left(2x-x^2\right)\cdot\left(2x^2-x^3\right)\ \mathrm{d}x \\
&= \int_0^2 \left(8x-14x^2+6x^3\right)\ \mathrm{d}x + \int_0^2 \left(8x^4-6x^6+2x^7\right)\ \mathrm{d}x \\
&= \left[4x^2-\frac{14}{3}x^3+\frac{3}{2}x^4\right]_0^2 + \left[\frac{8}{5}x^5-\frac{6}{7}x^7+\frac{1}{4}x^8\right]_0^2 \\
&= \frac{8}{3}+\frac{192}{35} \ = \ \frac{856}{105} \ \doteq \ 8.152
\end{aligned}$$

$$\begin{aligned}
(v_1, v_3)_A &= \int_0^2 (2-2x)\cdot\left(6x^2-4x^3\right)\ \mathrm{d}x + \int_0^2 \left(2x+2x^2\right)\cdot\left(2x-x^2\right)\cdot\left(2x^3-x^4\right)\ \mathrm{d}x \\
&= \int_0^2 \left(12x^2-20x^3+8x^4\right)\ \mathrm{d}x + \int_0^2 \left(8x^5-6x^7+2x^8\right)\ \mathrm{d}x \\
&= \left[4x^3-5x^4+\frac{8}{5}x^5\right]_0^2 + \left[\frac{4}{3}x^6-\frac{3}{4}x^8+\frac{2}{9}x^9\right]_0^2 \\
&= \frac{16}{5}+\frac{64}{9} \ = \ \frac{464}{45} \ \doteq \ 10.311
\end{aligned}$$

$$\begin{aligned}
(v_2, v_2)_A &= \int_0^2 \left(4x-3x^2\right)\cdot\left(4x-3x^2\right)\ \mathrm{d}x + \int_0^2 \left(2x+2x^2\right)\cdot\left(2x^2-x^3\right)\cdot\left(2x^2-x^3\right)\ \mathrm{d}x \\
&= \int_0^2 \left(16x^2-24x^3+9x^4\right)\ \mathrm{d}x + \int_0^2 \left(8x^5-6x^7+2x^8\right)\ \mathrm{d}x \\
&= \left[\frac{16}{3}x^3-6x^4+\frac{9}{5}x^5\right]_0^2 + \left[\frac{4}{3}x^6-\frac{3}{4}x^8+\frac{2}{9}x^9\right]_0^2 \\
&= \frac{64}{15}+\frac{64}{9} \ = \ \frac{512}{45} \ \doteq \ 11.378
\end{aligned}$$

$$\begin{aligned}
(v_2, v_3)_A &= \int_0^2 \left(4x-3x^2\right)\cdot\left(6x^2-4x^3\right)\ \mathrm{d}x + \int_0^2 \left(2x+2x^2\right)\cdot\left(2x^2-x^3\right)\cdot\left(2x^3-x^4\right)\ \mathrm{d}x \\
&= \int_0^2 \left(24x^3-34x^4+12x^5\right)\ \mathrm{d}x + \int_0^2 \left(8x^6-6x^8+2x^9\right)\ \mathrm{d}x \\
&= \left[6x^4-\frac{34}{5}x^5+2x^6\right]_0^2 + \left[\frac{8}{7}x^7-\frac{2}{3}x^9+\frac{1}{5}x^{10}\right]_0^2 \\
&= \frac{32}{5}+\frac{1024}{105} \ = \ \frac{1696}{105} \ \doteq \ 16.152
\end{aligned}$$

$$
\begin{aligned}
(v_3, v_3)_A &= \int_0^2 \left(6x^2 - 4x^3\right) \cdot \left(6x^2 - 4x^3\right) \, \mathrm{d}x + \int_0^2 \left(2x + 2x^2\right) \cdot \left(2x^3 - x^4\right) \cdot \left(2x^3 - x^4\right) \, \mathrm{d}x \\
&= \int_0^2 \left(36x^4 - 48x^5 + 16x^6\right) \, \mathrm{d}x + \int_0^2 \left(8x^7 - 6x^9 + 2x^{10}\right) \, \mathrm{d}x \\
&= \left[\frac{36}{5}x^5 - 8x^6 + \frac{16}{7}x^7\right]_0^2 + \left[x^8 - \frac{3}{5}x^{10} + \frac{2}{11}x^{11}\right]_0^2 \\
&= \frac{384}{35} + \frac{768}{55} \;=\; \frac{1920}{77} \doteq 24.935
\end{aligned}
$$

- koeficienty pravé strany soustavy rovnic:

$$
(f, v_1) = \int_0^2 (-3) \cdot \left(2x - x^2\right) \, \mathrm{d}x = \int_0^2 \left(-6x + 3x^2\right) \, \mathrm{d}x = \left[-3x^2 + x^3\right]_0^2 = -4
$$

$$
(f, v_2) = \int_0^2 (-3) \cdot \left(2x^2 - x^3\right) \, \mathrm{d}x = \int_0^2 \left(-6x^2 + 3x^3\right) \, \mathrm{d}x = \left[-2x^3 + \frac{3}{4}x^4\right]_0^2 = -4
$$

$$
(f, v_3) = \int_0^2 (-3) \cdot \left(2x^3 - x^4\right) \, \mathrm{d}x = \int_0^2 \left(-6x^3 + 3x^4\right) \, \mathrm{d}x = \left[-\frac{3}{2}x^4 + \frac{3}{5}x^5\right]_0^2 = -\frac{24}{5} \doteq -4.8
$$

- rozšířená matice soustavy rovnic:

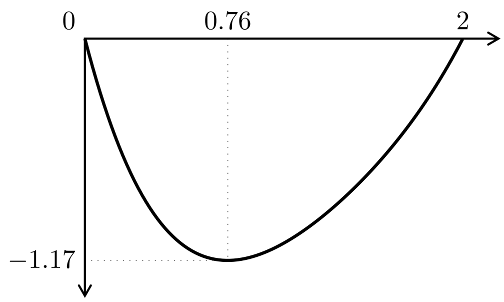
$$
\left(\begin{array}{ccc|c}
\frac{152}{21} & \frac{856}{105} & \frac{464}{45} & -4 \\
\frac{856}{105} & \frac{512}{45} & \frac{1696}{105} & -4 \\
\frac{464}{45} & \frac{1696}{105} & \frac{1920}{77} & -\frac{24}{5}
\end{array}\right)
\doteq
\left(\begin{array}{rrr|r}
7.238 & 8.152 & 10.311 & -4.0 \\
8.152 & 11.378 & 16.152 & -4.0 \\
10.311 & 16.152 & 24.935 & -4.8
\end{array}\right)
$$

- řešení soustavy rovnic:

$$
\alpha_1 \doteq -0.965, \qquad \alpha_2 \doteq 0.58, \qquad \alpha_3 \doteq -0.169
$$

- aproximace řešení:

$$
\begin{aligned}
\tilde{u}(x) &= -0.965 \cdot \left(2x - x^2\right) + 0.58 \cdot \left(2x^2 - x^3\right) - 0.169 \cdot \left(2x^3 - x^4\right) \\
&= -3.859x + 4.251x^2 - 1.838x^3 + 0.339x^4
\end{aligned}
$$



Úloha. Řešte okrajovou úlohu

$$
-u''(x) + (2 + 2x)\, u(x) = -2; \quad x \in \langle 0, 3 \rangle; \quad u(0) = 0, \;\; u(3) = 0,
$$

Ritzovou metodou s volbou báze

$$
v_1(x) = x(3 - x), \quad v_2(x) = x^2(3 - x), \quad v_3(x) = x^3(3 - x).
$$

Řešení.

- první derivace bázových funkcí:

$$\begin{aligned} v_1(x) &= 3x - x^2, & v_1'(x) &= 3 - 2x, \\ v_2(x) &= 3x^2 - x^3, & v_2'(x) &= 6x - 3x^2, \\ v_3(x) &= 3x^3 - x^4, & v_3'(x) &= 9x^2 - 4x^3. \end{aligned}$$

- energetický součin:

$$\begin{aligned} (u,v)_A &= \int_0^3 (-u''(x) + (2+2x)\,u(x))\,v(x)\,\mathrm{d}x \\ &= -\int_0^3 u''(x)v(x)\,\mathrm{d}x + \int_0^3 (2+2x)\,u(x)v(x)\,\mathrm{d}x \\ &= \int_0^3 u'(x)v'(x)\,\mathrm{d}x + \int_0^3 (2+2x)\,u(x)v(x)\,\mathrm{d}x \end{aligned}$$

- koeficienty matice soustavy rovnic:

$$\begin{aligned} (v_1,v_1)_A &= \int_0^3 (3-2x)\cdot(3-2x)\ \mathrm{d}x + \int_0^3 (2+2x)\cdot\left(3x-x^2\right)\cdot\left(3x-x^2\right)\ \mathrm{d}x \\ &= \int_0^3 \left(9-12x+4x^2\right)\ \mathrm{d}x + \int_0^3 \left(18x^2+6x^3-10x^4+2x^5\right)\ \mathrm{d}x \\ &= \left[9x-6x^2+\frac{4}{3}x^3\right]_0^3 + \left[6x^3+\frac{3}{2}x^4-2x^5+\frac{1}{3}x^6\right]_0^3 \\ &= 9+\frac{81}{2} = \frac{99}{2} \doteq 49.5 \end{aligned}$$

$$\begin{aligned} (v_1,v_2)_A &= \int_0^3 (3-2x)\cdot\left(6x-3x^2\right)\ \mathrm{d}x + \int_0^3 (2+2x)\cdot\left(3x-x^2\right)\cdot\left(3x^2-x^3\right)\ \mathrm{d}x \\ &= \int_0^3 \left(18x-21x^2+6x^3\right)\ \mathrm{d}x + \int_0^3 \left(18x^3+6x^4-10x^5+2x^6\right)\ \mathrm{d}x \\ &= \left[9x^2-7x^3+\frac{3}{2}x^4\right]_0^3 + \left[\frac{9}{2}x^4+\frac{6}{5}x^5-\frac{5}{3}x^6+\frac{2}{7}x^7\right]_0^3 \\ &= \frac{27}{2}+\frac{4617}{70} = \frac{2781}{35} \doteq 79.457 \end{aligned}$$

$$\begin{aligned} (v_1,v_3)_A &= \int_0^3 (3-2x)\cdot\left(9x^2-4x^3\right)\ \mathrm{d}x + \int_0^3 (2+2x)\cdot\left(3x-x^2\right)\cdot\left(3x^3-x^4\right)\ \mathrm{d}x \\ &= \int_0^3 \left(27x^2-30x^3+8x^4\right)\ \mathrm{d}x + \int_0^3 \left(18x^4+6x^5-10x^6+2x^7\right)\ \mathrm{d}x \\ &= \left[9x^3-\frac{15}{2}x^4+\frac{8}{5}x^5\right]_0^3 + \left[\frac{18}{5}x^5+x^6-\frac{10}{7}x^7+\frac{1}{4}x^8\right]_0^3 \\ &= \frac{243}{10}+\frac{16767}{140} = \frac{20169}{140} \doteq 144.064 \end{aligned}$$

$$\begin{aligned} (v_2,v_2)_A &= \int_0^3 \left(6x-3x^2\right)\cdot\left(6x-3x^2\right)\ \mathrm{d}x + \int_0^3 (2+2x)\cdot\left(3x^2-x^3\right)\cdot\left(3x^2-x^3\right)\ \mathrm{d}x \\ &= \int_0^3 \left(36x^2-36x^3+9x^4\right)\ \mathrm{d}x + \int_0^3 \left(18x^4+6x^5-10x^6+2x^7\right)\ \mathrm{d}x \\ &= \left[12x^3-9x^4+\frac{9}{5}x^5\right]_0^3 + \left[\frac{18}{5}x^5+x^6-\frac{10}{7}x^7+\frac{1}{4}x^8\right]_0^3 \\ &= \frac{162}{5}+\frac{16767}{140} = \frac{21303}{140} \doteq 152.164 \end{aligned}$$

$$
\begin{aligned}
(v_2,v_3)_A &= \int_0^3 \left(6x-3x^2\right)\cdot\left(9x^2-4x^3\right)\,\mathrm{d}x + \int_0^3 (2+2x)\cdot\left(3x^2-x^3\right)\cdot\left(3x^3-x^4\right)\,\mathrm{d}x \\
&= \int_0^3 \left(54x^3-51x^4+12x^5\right)\,\mathrm{d}x + \int_0^3 \left(18x^5+6x^6-10x^7+2x^8\right)\,\mathrm{d}x \\
&= \left[\frac{27}{2}x^4-\frac{51}{5}x^5+2x^6\right]_0^3 + \left[3x^6+\frac{6}{7}x^7-\frac{5}{4}x^8+\frac{2}{9}x^9\right]_0^3 \\
&= \frac{729}{10}+\frac{6561}{28} \;=\; \frac{43011}{140} \;\doteq\; 307.221
\end{aligned}
$$

$$
\begin{aligned}
(v_3,v_3)_A &= \int_0^3 \left(9x^2-4x^3\right)\cdot\left(9x^2-4x^3\right)\,\mathrm{d}x + \int_0^3 (2+2x)\cdot\left(3x^3-x^4\right)\cdot\left(3x^3-x^4\right)\,\mathrm{d}x \\
&= \int_0^3 \left(81x^4-72x^5+16x^6\right)\,\mathrm{d}x + \int_0^3 \left(18x^6+6x^7-10x^8+2x^9\right)\,\mathrm{d}x \\
&= \left[\frac{81}{5}x^5-12x^6+\frac{16}{7}x^7\right]_0^3 + \left[\frac{18}{7}x^7+\frac{3}{4}x^8-\frac{10}{9}x^9+\frac{1}{5}x^{10}\right]_0^3 \\
&= \frac{6561}{35}+\frac{67797}{140} \;=\; \frac{94041}{140} \;\doteq\; 671.721
\end{aligned}
$$

- koeficienty pravé strany soustavy rovnic:

$$
(f,v_1) \;=\; \int_0^3 (-2)\cdot\left(3x-x^2\right)\,\mathrm{d}x \;=\; \int_0^3 \left(-6x+2x^2\right)\,\mathrm{d}x \;=\; \left[-3x^2+\frac{2}{3}x^3\right]_0^3 \;=\; -9
$$

$$
(f,v_2) \;=\; \int_0^3 (-2)\cdot\left(3x^2-x^3\right)\,\mathrm{d}x \;=\; \int_0^3 \left(-6x^2+2x^3\right)\,\mathrm{d}x \;=\; \left[-2x^3+\frac{1}{2}x^4\right]_0^3 \;=\; -\frac{27}{2} \;=\; -13.5
$$

$$
(f,v_3) \;=\; \int_0^3 (-2)\cdot\left(3x^3-x^4\right)\,\mathrm{d}x \;=\; \int_0^3 \left(-6x^3+2x^4\right)\,\mathrm{d}x \;=\; \left[-\frac{3}{2}x^4+\frac{2}{5}x^5\right]_0^3 \;=\; -\frac{243}{10} \;=\; -24.3
$$

- rozšířená matice soustavy rovnic:

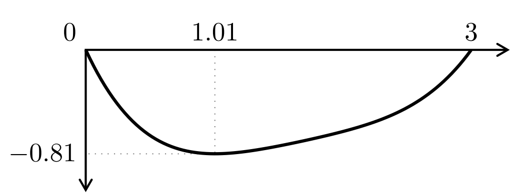
$$
\left(\begin{array}{ccc|c}
\frac{99}{2} & \frac{2781}{35} & \frac{20169}{140} & -9 \\
\frac{2781}{35} & \frac{21303}{140} & \frac{43011}{140} & -\frac{27}{2} \\
\frac{20169}{140} & \frac{43011}{140} & \frac{94041}{140} & -\frac{243}{10}
\end{array}\right)
\;\doteq\;
\left(\begin{array}{ccc|c}
49.5 & 79.457 & 144.064 & -9.0 \\
79.457 & 152.164 & 307.221 & -13.5 \\
144.064 & 307.221 & 671.721 & -24.3
\end{array}\right)
$$

- řešení soustavy rovnic:

$$
\alpha_1 \;\doteq\; -0.36, \qquad \alpha_2 \;\doteq\; 0.214, \qquad \alpha_3 \;\doteq\; -0.057
$$

- aproximace řešení:

$$
\begin{aligned}
\tilde{u}(x) &= -0.36\cdot\left(3x-x^2\right)+0.214\cdot\left(3x^2-x^3\right)-0.057\cdot\left(3x^3-x^4\right) \\
&= -2.161x+2.007x^2-0.771x^3+0.114x^4
\end{aligned}
$$



Úloha. Řešte okrajovou úlohu

$$
-u''(x)+\left(x-2x^2\right)u(x)=x; \quad x\in\langle 0,3\rangle\,; \quad u(0)=0,\; u(3)=0,
$$

Ritzovou metodou s volbou báze

$$
v_1(x) \;=\; x(3-x), \quad v_2(x) \;=\; x^2(3-x), \quad v_3(x) \;=\; x^3(3-x), \quad v_4(x) \;=\; x^4(3-x).
$$

Řešení.

- první derivace bázových funkcí:

$$\begin{aligned} v_1(x) &= 3x - x^2, & v_1'(x) &= 3 - 2x, \\ v_2(x) &= 3x^2 - x^3, & v_2'(x) &= 6x - 3x^2, \\ v_3(x) &= 3x^3 - x^4, & v_3'(x) &= 9x^2 - 4x^3, \\ v_4(x) &= 3x^4 - x^5, & v_4'(x) &= 12x^3 - 5x^4. \end{aligned}$$

- energetický součin:

$$\begin{aligned} (u,v)_A &= \int_0^3 \left(-u''(x) + \left(x - 2x^2\right) u(x)\right) v(x)\,\mathrm{d}x \\ &= -\int_0^3 u''(x)v(x)\,\mathrm{d}x + \int_0^3 \left(x - 2x^2\right) u(x)v(x)\,\mathrm{d}x \\ &= \int_0^3 u'(x)v'(x)\,\mathrm{d}x + \int_0^3 \left(x - 2x^2\right) u(x)v(x)\,\mathrm{d}x \end{aligned}$$

- koeficienty matice soustavy rovnic:

$$\begin{aligned} (v_1,v_1)_A &= \int_0^3 (3-2x)\cdot(3-2x)\ \mathrm{d}x + \int_0^3 \left(x-2x^2\right)\cdot\left(3x-x^2\right)\cdot\left(3x-x^2\right)\ \mathrm{d}x \\ &= \int_0^3 \left(9 - 12x + 4x^2\right)\ \mathrm{d}x + \int_0^3 \left(9x^3 - 24x^4 + 13x^5 - 2x^6\right)\ \mathrm{d}x \\ &= \left[9x - 6x^2 + \frac{4}{3}x^3\right]_0^3 + \left[\frac{9}{4}x^4 - \frac{24}{5}x^5 + \frac{13}{6}x^6 - \frac{2}{7}x^7\right]_0^3 \\ &= 9 + -\frac{4131}{140} \ = \ -\frac{2871}{140} \ \doteq \ -20.507 \end{aligned}$$

$$\begin{aligned} (v_1,v_2)_A &= \int_0^3 (3-2x)\cdot\left(6x-3x^2\right)\ \mathrm{d}x + \int_0^3 \left(x-2x^2\right)\cdot\left(3x-x^2\right)\cdot\left(3x^2-x^3\right)\ \mathrm{d}x \\ &= \int_0^3 \left(18x - 21x^2 + 6x^3\right)\ \mathrm{d}x + \int_0^3 \left(9x^4 - 24x^5 + 13x^6 - 2x^7\right)\ \mathrm{d}x \\ &= \left[9x^2 - 7x^3 + \frac{3}{2}x^4\right]_0^3 + \left[\frac{9}{5}x^5 - 4x^6 + \frac{13}{7}x^7 - \frac{1}{4}x^8\right]_0^3 \\ &= \frac{27}{2} + -\frac{8019}{140} \ = \ -\frac{6129}{140} \ \doteq \ -43.779 \end{aligned}$$

$$\begin{aligned} (v_1,v_3)_A &= \int_0^3 (3-2x)\cdot\left(9x^2-4x^3\right)\ \mathrm{d}x + \int_0^3 \left(x-2x^2\right)\cdot\left(3x-x^2\right)\cdot\left(3x^3-x^4\right)\ \mathrm{d}x \\ &= \int_0^3 \left(27x^2 - 30x^3 + 8x^4\right)\ \mathrm{d}x + \int_0^3 \left(9x^5 - 24x^6 + 13x^7 - 2x^8\right)\ \mathrm{d}x \\ &= \left[9x^3 - \frac{15}{2}x^4 + \frac{8}{5}x^5\right]_0^3 + \left[\frac{3}{2}x^6 - \frac{24}{7}x^7 + \frac{13}{8}x^8 - \frac{2}{9}x^9\right]_0^3 \\ &= \frac{243}{10} + -\frac{6561}{56} \ = \ -\frac{26001}{280} \ \doteq \ -92.861 \end{aligned}$$

$$\begin{aligned} (v_1,v_4)_A &= \int_0^3 (3-2x)\cdot\left(12x^3-5x^4\right)\ \mathrm{d}x + \int_0^3 \left(x-2x^2\right)\cdot\left(3x-x^2\right)\cdot\left(3x^4-x^5\right)\ \mathrm{d}x \\ &= \int_0^3 \left(36x^3 - 39x^4 + 10x^5\right)\ \mathrm{d}x + \int_0^3 \left(9x^6 - 24x^7 + 13x^8 - 2x^9\right)\ \mathrm{d}x \\ &= \left[9x^4 - \frac{39}{5}x^5 + \frac{5}{3}x^6\right]_0^3 + \left[\frac{9}{7}x^7 - 3x^8 + \frac{13}{9}x^9 - \frac{1}{5}x^{10}\right]_0^3 \\ &= \frac{243}{5} + -\frac{8748}{35} \ = \ -\frac{7047}{35} \ \doteq \ -201.343 \end{aligned}$$

$$
\begin{aligned}
(v_2, v_2)_A &= \int_0^3 \left(6x - 3x^2\right) \cdot \left(6x - 3x^2\right) \, \mathrm{d}x + \int_0^3 \left(x - 2x^2\right) \cdot \left(3x^2 - x^3\right) \cdot \left(3x^2 - x^3\right) \, \mathrm{d}x \\
&= \int_0^3 \left(36x^2 - 36x^3 + 9x^4\right) \, \mathrm{d}x + \int_0^3 \left(9x^5 - 24x^6 + 13x^7 - 2x^8\right) \, \mathrm{d}x \\
&= \left[12x^3 - 9x^4 + \frac{9}{5}x^5\right]_0^3 + \left[\frac{3}{2}x^6 - \frac{24}{7}x^7 + \frac{13}{8}x^8 - \frac{2}{9}x^9\right]_0^3 \\
&= \frac{162}{5} + -\frac{6561}{56} \quad = \quad -\frac{23733}{280} \;\doteq\; -84.761
\end{aligned}
$$

$$
\begin{aligned}
(v_2, v_3)_A &= \int_0^3 \left(6x - 3x^2\right) \cdot \left(9x^2 - 4x^3\right) \, \mathrm{d}x + \int_0^3 \left(x - 2x^2\right) \cdot \left(3x^2 - x^3\right) \cdot \left(3x^3 - x^4\right) \, \mathrm{d}x \\
&= \int_0^3 \left(54x^3 - 51x^4 + 12x^5\right) \, \mathrm{d}x + \int_0^3 \left(9x^6 - 24x^7 + 13x^8 - 2x^9\right) \, \mathrm{d}x \\
&= \left[\frac{27}{2}x^4 - \frac{51}{5}x^5 + 2x^6\right]_0^3 + \left[\frac{9}{7}x^7 - 3x^8 + \frac{13}{9}x^9 - \frac{1}{5}x^{10}\right]_0^3 \\
&= \frac{729}{10} + -\frac{8748}{35} \quad = \quad -\frac{12393}{70} \;\doteq\; -177.043
\end{aligned}
$$

$$
\begin{aligned}
(v_2, v_4)_A &= \int_0^3 \left(6x - 3x^2\right) \cdot \left(12x^3 - 5x^4\right) \, \mathrm{d}x + \int_0^3 \left(x - 2x^2\right) \cdot \left(3x^2 - x^3\right) \cdot \left(3x^4 - x^5\right) \, \mathrm{d}x \\
&= \int_0^3 \left(72x^4 - 66x^5 + 15x^6\right) \, \mathrm{d}x + \int_0^3 \left(9x^7 - 24x^8 + 13x^9 - 2x^{10}\right) \, \mathrm{d}x \\
&= \left[\frac{72}{5}x^5 - 11x^6 + \frac{15}{7}x^7\right]_0^3 + \left[\frac{9}{8}x^8 - \frac{8}{3}x^9 + \frac{13}{10}x^{10} - \frac{2}{11}x^{11}\right]_0^3 \\
&= \frac{5832}{35} + -\frac{242757}{440} \quad = \quad -\frac{1186083}{3080} \;\doteq\; -385.092
\end{aligned}
$$

$$
\begin{aligned}
(v_3, v_3)_A &= \int_0^3 \left(9x^2 - 4x^3\right) \cdot \left(9x^2 - 4x^3\right) \, \mathrm{d}x + \int_0^3 \left(x - 2x^2\right) \cdot \left(3x^3 - x^4\right) \cdot \left(3x^3 - x^4\right) \, \mathrm{d}x \\
&= \int_0^3 \left(81x^4 - 72x^5 + 16x^6\right) \, \mathrm{d}x + \int_0^3 \left(9x^7 - 24x^8 + 13x^9 - 2x^{10}\right) \, \mathrm{d}x \\
&= \left[\frac{81}{5}x^5 - 12x^6 + \frac{16}{7}x^7\right]_0^3 + \left[\frac{9}{8}x^8 - \frac{8}{3}x^9 + \frac{13}{10}x^{10} - \frac{2}{11}x^{11}\right]_0^3 \\
&= \frac{6561}{35} + -\frac{242757}{440} \quad = \quad -\frac{1121931}{3080} \;\doteq\; -364.263
\end{aligned}
$$

$$
\begin{aligned}
(v_3, v_4)_A &= \int_0^3 \left(9x^2 - 4x^3\right) \cdot \left(12x^3 - 5x^4\right) \, \mathrm{d}x + \int_0^3 \left(x - 2x^2\right) \cdot \left(3x^3 - x^4\right) \cdot \left(3x^4 - x^5\right) \, \mathrm{d}x \\
&= \int_0^3 \left(108x^5 - 93x^6 + 20x^7\right) \, \mathrm{d}x + \int_0^3 \left(9x^8 - 24x^9 + 13x^{10} - 2x^{11}\right) \, \mathrm{d}x \\
&= \left[18x^6 - \frac{93}{7}x^7 + \frac{5}{2}x^8\right]_0^3 + \left[x^9 - \frac{12}{5}x^{10} + \frac{13}{11}x^{11} - \frac{1}{6}x^{12}\right]_0^3 \\
&= \frac{6561}{14} + -\frac{137781}{110} \quad = \quad -\frac{301806}{385} \;\doteq\; -783.912
\end{aligned}
$$

$$
\begin{aligned}
(v_4, v_4)_A &= \int_0^3 \left(12x^3 - 5x^4\right) \cdot \left(12x^3 - 5x^4\right) \, \mathrm{d}x + \int_0^3 \left(x - 2x^2\right) \cdot \left(3x^4 - x^5\right) \cdot \left(3x^4 - x^5\right) \, \mathrm{d}x \\
&= \int_0^3 \left(144x^6 - 120x^7 + 25x^8\right) \, \mathrm{d}x + \int_0^3 \left(9x^9 - 24x^{10} + 13x^{11} - 2x^{12}\right) \, \mathrm{d}x \\
&= \left[\frac{144}{7}x^7 - 15x^8 + \frac{25}{9}x^9\right]_0^3 + \left[\frac{9}{10}x^{10} - \frac{24}{11}x^{11} + \frac{13}{12}x^{12} - \frac{2}{13}x^{13}\right]_0^3 \\
&= \frac{8748}{7} + -\frac{8325909}{2860} \quad = \quad -\frac{33262083}{20020} \;\doteq\; -1661.443
\end{aligned}
$$

- koeficienty pravé strany soustavy rovnic:

$$
\begin{aligned}
(f,v_1) &= \int_0^3 (x)\cdot\left(3x-x^2\right)\,\mathrm{d}x = \int_0^3 \left(3x^2-x^3\right)\,\mathrm{d}x = \left[x^3-\frac{1}{4}x^4\right]_0^3 = \frac{27}{4} \doteq 6.75\\
(f,v_2) &= \int_0^3 (x)\cdot\left(3x^2-x^3\right)\,\mathrm{d}x = \int_0^3 \left(3x^3-x^4\right)\,\mathrm{d}x = \left[\frac{3}{4}x^4-\frac{1}{5}x^5\right]_0^3 = \frac{243}{20} \doteq 12.15\\
(f,v_3) &= \int_0^3 (x)\cdot\left(3x^3-x^4\right)\,\mathrm{d}x = \int_0^3 \left(3x^4-x^5\right)\,\mathrm{d}x = \left[\frac{3}{5}x^5-\frac{1}{6}x^6\right]_0^3 = \frac{243}{10} \doteq 24.3\\
(f,v_4) &= \int_0^3 (x)\cdot\left(3x^4-x^5\right)\,\mathrm{d}x = \int_0^3 \left(3x^5-x^6\right)\,\mathrm{d}x = \left[\frac{1}{2}x^6-\frac{1}{7}x^7\right]_0^3 = \frac{729}{14} \doteq 52.071
\end{aligned}
$$

- rozšířená matice soustavy rovnic:

$$
\left(\begin{array}{cccc|c}
-20.507 & -43.779 & -92.861 & -201.343 & 6.75\\
-43.779 & -84.761 & -177.043 & -385.092 & 12.15\\
-92.861 & -177.043 & -364.263 & -783.912 & 24.3\\
-201.343 & -385.092 & -783.912 & -1661.443 & 52.071
\end{array}\right)
$$

- řešení soustavy rovnic:

$$
\alpha_1 \doteq 0.007, \quad \alpha_2 \doteq 0.466, \quad \alpha_3 \doteq -0.438, \quad \alpha_4 \doteq 0.066
$$

- aproximace řešení:

$$
\begin{aligned}
\tilde{u}(x) &= 0.007\cdot\left(3x-x^2\right)+0.466\cdot\left(3x^2-x^3\right)-0.438\cdot\left(3x^3-x^4\right)+0.066\cdot\left(3x^4-x^5\right)\\
&= 0.044x+2.782x^2-3.558x^3+1.272x^4-0.132x^5
\end{aligned}
$$