Dominance on continuous Archimedean triangular norms and generalized Mulholland inequality

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February 5, 2020

Abstract

As a preceding result, it has been shown that the dominance relation is not transitive on the set of strict triangular norms. This result has been achieved thanks to new results on Mulholland inequality.

Recently, Saminger-Platz, De Baets, and De Meyer have introduced the generalized Mulholland inequality which characterizes the dominance on all continuous Archimedean triangular norms in an analogous way as does Mulholland inequality on the strict triangular norms. Based on these new results, the present paper shows that the dominance relation is not transitive on the set of nilpotent triangular norms and, consequently, on the set of continuous Archimedean triangular norms.

This result is achieved by introducing a new sufficient condition under which a given function solves the generalized Mulholland inequality and by showing that the set of the functions that solve the inequality is not closed with respect to compositions.

Keywords: *dominance relation, generalized Mulholland inequality, nilpotent triangular norm, transitivity.*
MSC: *Primary 26D07; Secondary 39B72, 26D15, 26A51, 03E72, 54E70.*

1 Introduction

Mulholland inequality is a functional inequality which has been introduced in a paper by Mulholland [11] in 1947 as a generalization of Minkowski inequality which describes the triangular inequality of L^p-norms. In Minkowski inequality, Mulholland has replaced the power functions $x \mapsto x^p$, $p > 1$, by an arbitrary increasing bijection obtaining a generalization.

Few decades later, in 1984, Tardiff has revealed a close connection between Mulholland inequality and the dominance relation defined on the set of triangular norms [22] (shortly t-norms). He has shown that Mulholland inequality directly characterizes the dominance on a subset of t-norms called strict t-norms.

Recall that both the notions of dominance and t-norms [2, 8] have been introduced within the framework of probabilistic metric spaces [10, 19]; t-norms describe the triangular inequality of the probabilistic metrics while a satisfaction of the dominance relation is crucial when constructing Cartesian products of probabilistic metric spaces [21]. Later on, t-norms have found their role also as the interpretation of the logical conjunction in the semantics of fuzzy logics [4, 5, 6, 12]. Recall that the dominance has found its use also when working with t-norm based fuzzy equivalences and partitions [3, Theorem 2] and with their refinements [3, Theorem 5].

It is easy to show that the relation of dominance is both reflexive and antisymmetric. However, for a long time the question, whether this relation is also transitive, and thus an order, was open. This question has been even stated as an open problem in the monograph by Schweizer and Sklar [19, Problem 12.11.3] as well as in the list of open problems by Alsina, Frank, and Schweizer [1, Problem 17]. Recently, in 2008, it has been answered negatively for the class of continuous t-norms in a paper by Sarkoci [18]. A few years later, a negative answer has been given also for strict t-norms [14]. This answer has been achieved thanks to new results on Mulholland inequality [13]. Remark that the question has remained open for the class of nilpotent t-norms, which also form an important sub-class of continuous t-norms, although some parametric sub-classes of this class have been already studied [7, 8, 15, 20].

In 2008, Saminger-Platz, De Baets, and De Meyer have introduced the generalized Mulholland inequality [17] which, analogously to the original Mulholland inequality, characterizes the dominance on the class of all continuous Archimedean t-norms, i.e., on both strict and nilpotent t-norms. It is therefore natural to ask whether this generalized inequality can help to answer the question of the transitivity of the dominance also for nilpotent t-norms and, generally, for all continuous Archimedean t-norms.

This research follows up on the latter cited paper stating some new results on the generalized Mulholland inequality. Further, as a corollary, the question of the transitivity of the dominance on nilpotent t-norms is answered. The paper is similar to previous ones [14, 13] stating analogous results which have been already made for the original Mulholland inequality and giving analogous proofs. However, as the assumptions are different to the previous case, it has shown as a better option to write all the proofs again. Moreover, despite the analogous approach we end up this time with a different solution—an answer for the case of nilpotent t-norms.

2 Triangular norms and dominance relation

A *triangular norm* [2, 8] (shortly a *t-norm*) is a commutative, associative, and monotone binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ with a neutral element 1, i.e., we have $x*1=1*x=x$ for every $x\in[0,1]$.

For $n\in\mathbb{N}$, we define the n-th natural power of $x\in[0,1]$, according to a t-norm $*$, by $x_*^{(n)}=x_*^{(n-1)}*x$ and $x_*^{(0)}=1$. A t-norm $*$ is said to be *continuous* if it is continuous as a two-variable real function. It is said to be *Archimedean* if, for every $x,y\in\,]0,1[$, such that $x<y$, there is $n\in\mathbb{N}$ such that $y_*^{(n)}<x$. A continuous t-norm $*$ is Archimedean if, and only if, $x*x<x$ for every $x\in\,]0,1[$.

A continuous t-norm $*$ is said to be *nilpotent* if, for every $x\in\,]0,1[$ there is $n\in\mathbb{N}$ such that $x_*^{(n)}=0$. It is said to be *strict* if its restriction to $]0,1]^2$ is strictly increasing in each variable. A continuous Archimedean t-norm is either nilpotent or strict.

By $[0,\infty]$ we denote the interval of non-negative real numbers enriched by the top element ∞ with $x+\infty=\infty$ for any $x\in[0,\infty]$.

A t-norm $*$ is continuous Archimedean if, and only if, there is a continuous decreasing injection $t\colon[0,1]\to[0,\infty]$ with $t(1)=0$ such that

$$x*y \;=\; t^{(-1)}\big(t(x)+t(y)\big)$$

for every $x,y\in[0,1]$. Here, $t^{(-1)}$ denotes the *pseudo-inverse* of t defined, in this particular case, by $t^{(-1)}(x)=t^{-1}(x)$ if $x\le t(0)$ and $t^{(-1)}(x)=0$ otherwise. The mapping t is called the *additive generator* of the continuous Archimedean t-norm $*$ and this generator is unique up to a multiplication by a positive real constant. The t-norm generated by an additive generator t is nilpotent if $t(0)<\infty$; if $t(0)=\infty$ then $*$ is strict.

Dominance is a binary relation defined on the set of t-norms by Tardiff [21, Definition 3.4]. Remark that this relation can be defined also in a more general setting [16]. A t-norm $*_1$ is said to *dominate* a t-norm $*_2$ (and we write $*_1\gg *_2$) if

$$\forall x,y,u,v\in[0,1]:\quad (x*_2 y)*_1(u*_2 v)\quad\ge\quad (x*_1 u)*_2(y*_1 v). \qquad (1)$$

By setting $y=u=1$ in (1), we have that $*_1\gg *_2$ implies $*_1\ge *_2$ for every two t-norms $*_1$ and $*_2$. From this fact it follows that dominance is an anti-symmetric relation. Dominance is, furthermore, also reflexive since for any t-norm $*$ we have $*\gg *$ from the associativity and commutativity of $*$. In this paper, we are going to deal with the question whether, on certain classes of t-norms, the relation of dominance is also transitive, i.e., whether

$$*_1\gg *_2 \quad\text{and}\quad *_2\gg *_3 \quad\text{implies}\quad *_1\gg *_3$$

for every three t-norms $*_1$, $*_2$, and $*_3$ from a given class.

3 Generalized Mulholland inequality

An increasing bijection $f\colon [0,\infty[\to [0,\infty[$ is said to *solve Mulholland inequality* if

$$f^{-1}\left(\sum_{i=1}^{n} f\left(x_i+y_i\right)\right) \quad\leq\quad f^{-1}\left(\sum_{i=1}^{n} f\left(x_i\right)\right)+f^{-1}\left(\sum_{i=1}^{n} f\left(y_i\right)\right)$$

holds for every $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in [0,\infty[^n$. In this paper we will be exclusively using the two-dimensional ($n = 2$) variant of Mulholland inequality, i.e., f solves Mulholland inequality if

$$\begin{aligned}&\forall x, y, u, v \in [0,\infty[\,:\\ &f^{-1}\Big(f(x+u)+f(y+v)\Big) \leq f^{-1}\Big(f(x)+f(y)\Big)+f^{-1}\Big(f(u)+f(v)\Big).\end{aligned} \tag{MI}$$

Mulholland inequality is in a close correspondence with the dominance of strict t-norms which has been shown in 1984 by Tardiff [22, Theorem 3].

Notation 3.1 *The notation $f \circ g$ denotes the composition of two functions, f and g, such that it holds $(f \circ g)(x) = f(g(x))$ for every x in the domain of g.*

Theorem 3.2 *[22] Let $*_1$ and $*_2$ be two strict t-norms defined respectively by their additive generators t_1 and t_2, for every $x, y \in [0,1]$, as*

$$\begin{aligned} x *_1 y &= t_1^{-1}\big(t_1(x)+t_1(y)\big),\\ x *_2 y &= t_2^{-1}\big(t_2(x)+t_2(y)\big).\end{aligned}$$

Then $*_1$ dominates $*_2$ if, and only if, $f = t_1 \circ t_2^{-1}$ restricted to $[0,\infty[$ solves Mulholland inequality.

In 2008, Saminger-Platz, De Baets, and De Meyer have enlarged this correspondence to the set of all continuous Archimedean t-norms introducing the following result [17, Theorem 1]:

Theorem 3.3 *[17] Let $*_1$ and $*_2$ be two continuous Archimedean t-norms defined respectively by their additive generators t_1 and t_2, for every $x, y \in [0,1]$, as*

$$\begin{aligned} x *_1 y &= t_1^{(-1)}\big(t_1(x)+t_1(y)\big),\\ x *_2 y &= t_2^{(-1)}\big(t_2(x)+t_2(y)\big).\end{aligned}$$

Then $*_1$ dominates $*_2$ if, and only if, the functions $f = t_1 \circ t_2^{(-1)}$ and $f^{(-1)} = t_1 \circ t_2^{(-1)}$ satisfy

$$\begin{aligned}&\forall x, y, u, v \in [0, t_2(0)]:\\ &f^{(-1)}\Big(f(x+y)+f(u+v)\Big) \leq f^{(-1)}\Big(f(x)+f(u)\Big)+f^{(-1)}\Big(f(y)+f(v)\Big).\end{aligned} \tag{GMI}$$

If (GMI) holds we say that f *solves the generalized Mulholland inequality.* Analogously to the case of strict t-norms and Mulholland inequality, the generalized Mulholland inequality gives a characterization of the dominance on all continuous Archimedean t-norms which involve both strict and nilpotent t-norms. Notice that if we deal with strict t-norms and thus $t_1(0) = t_2(0) = \infty$ then (GMI) becomes equivalent to (MI). As the case of Mulholland inequality has been already described in the previous papers [13, 14], in this paper we will mostly focus on the case when $t_1(0) < \infty$ and $t_2(0) < \infty$, which describes the dominance on nilpotent t-norms.

We can see that, according to the properties of additive generators, f and $f^{(-1)}$ in Theorem 3.3 satisfy the following.

Assumptions 3.4 *Assume a function* $f\colon [0,\infty] \to [0,\infty]$ *and fixed values* $d, e \in]0,\infty]$ *such that:*

1. $f(0) = 0$ *and* $f(d) = e$*,*
2. f *is continuous and strictly increasing on* $[0, d]$*,*
3. $f(x) = e$ *for* $x \geq d$*.*

Assume, further, the function $f^{(-1)}\colon [0,\infty] \to [0,\infty]$ *defined by*

$$f^{(-1)}\colon \quad x \mapsto \begin{cases} f^{-1}(x) & \text{if } x \in [0,e]\,, \\ d & \text{otherwise.} \end{cases}$$

Note that if $f = t_1 \circ t_2^{(-1)}$ and $f^{(-1)} = t_1 \circ t_2^{(-1)}$ then we have $d = t_2(0)$ and $e = t_1(0)$.

The same paper, which has introduced the generalized Mulholland inequality [17], has presented also sufficient conditions under which a given function solves this inequality. One of them [17, Theorem 6] we are going to present here. According to the terminology of Matkowski [9], we define a function $f\colon [0,\infty] \to [0,\infty]$ to be *geometrically convex on an interval* $I \subseteq [0,\infty]$ if

$$f\left(x^{1-\alpha} \cdot y^{\alpha}\right) \quad \leq \quad f^{1-\alpha}(x) \cdot f^{\alpha}(y)$$

for every $x, y \in I$ and $\alpha \in [0, 1]$.

Theorem 3.5 *[17] Consider functions* f *and* $f^{(-1)}$*, which comply with Assumptions 3.4, such that* f *is*

- *convex on* $]0, d[$*,*
- *geometrically convex on* $]0, d[$*.*

Then f *solves the generalized Mulholland inequality.*

Note that f is geometrically convex on on $]0, d[$ if, and only if, the function $F = \log \circ f \circ \exp$, is convex on $]-\infty, \log d[$. We finish this section by one more result from the mentioned paper [17, Proposition 10]:

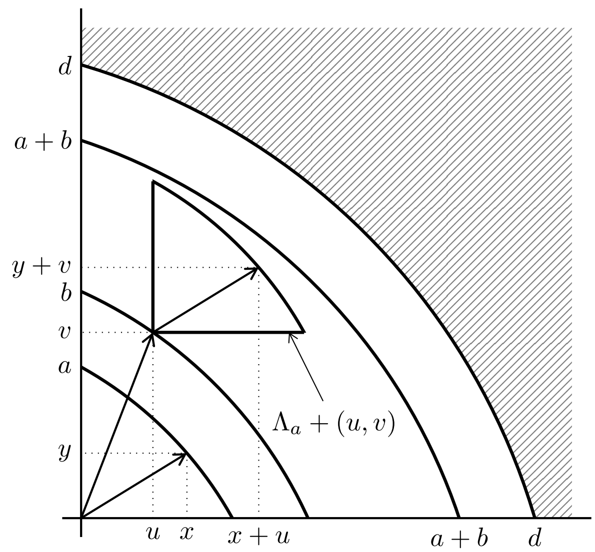

Figure 1: Geometric interpretation of the generalized Mulholland inequality.

Proposition 3.6 *[17] Consider functions f and $f^{(-1)}$, which comply with Assumptions 3.4. If f solves the generalized Mulholland inequality then it is convex on* $]0, d[$.

Remark that if f is convex resp. geometrically convex on $]0, d[$ then, since it is continuous, it is convex resp. geometrically convex also on $[0, d]$.

4 Bounded pseudo-addition

Throughout this section, f and $f^{(-1)}$ are given according to Assumptions 3.4 and f is assumed to be convex on $[0, d]$. Let $\oplus_f$ be a binary operation on $[0, \infty]$ defined by

$$x \oplus_f y \quad = \quad f^{(-1)}\Big(f(x) + f(y)\Big) \tag{2}$$

for every $x, y \in [0, \infty]$. We call $\oplus_f$ the *bounded pseudo-addition* generated by f. For $z \in [0, \infty]$, we define the *z-level set* of $\oplus_f$ as the set

$$L_z^f = \{(x, y) \in [0, \infty]^2 \mid x \oplus_f y = z\}$$

and the *z-level cut* of $\oplus_f$ as the set

$$\Lambda_z^f = \{(x, y) \in [0, \infty]^2 \mid x \oplus_f y \leq z\}.$$

Note that $(x, y) \in \Lambda_z^f$ if, and only if, $f(x) + f(y) \leq f(z)$. Finally, we define the *support of* $\oplus_f$ as the set

$$\text{supp} \oplus_f = \{(x, y) \in [0, \infty]^2 \mid x \oplus_f y < d\}.$$

In this paper we will often omit the the index f in the notation of $\oplus_f$, L_z^f, and Λ_z^f as it will be clear from the context which function f we are dealing with.

We can observe the following properties of $\oplus$.

- The operation is continuous, commutative, and associative.
- We have $x \oplus 0 = 0 \oplus x = x$ for every $x \in [0, d]$.
- The operation is strictly increasing in both variables if its domain is restricted to its support. On the rest of the domain, the operation $\oplus$ is the constant d.
- We have $x \oplus y \leq d$ for every $x, y \in [0, \infty]$; this gives the reason for "bounded" in the name of $\oplus$. (Be, however, aware that we can also have $d = \infty$; in such a case $x \oplus y$ is not "bounded".)

We call the collection of all the level sets (and the level cuts) of $\oplus$ the *level set plot of* $\oplus$. We can observe the following structure of such level set plots.

- For $z < d$, Λ_z is a convex set and L_z is its border in $[0, \infty]^2$.
- For $z = d$, $\Lambda_z = [0, \infty]^2$ and L_z is the complement of $\operatorname{supp} \oplus$.
- For $z > d$, $\Lambda_z = [0, \infty]^2$ and L_z is an empty set.

The operation of pseudo-addition allows us to rewrite (GMI) to the form:

$$\forall x, y, u, v \in [0, d]: \quad (x + u) \oplus (y + v) \ \leq \ (x \oplus y) + (u \oplus v). \qquad (\mathrm{GMI}_\oplus)$$

Observe that we have actually obtained the dominance inequality; thus a function f solves (GMI) if, and only if, the standard addition dominates the bounded pseudo-addition generated by f. Furthermore, ($\mathrm{GMI}_\oplus$) can be also understood as sub-additivity of $\oplus$.

Minkowski sum of $A, B \in [0, \infty]^2$ is defined by

$$A + B \ = \ \{(x + u, y + v) \mid (x, y) \in A, (u, v) \in B\}.$$

Theorem 4.1 *Let f and f^{-1} be two functions given according to Assumptions 3.4. Then they satisfy the generalized Mulholland inequality if, and only if,*

$$\forall a, b \in [0, \infty], \ a + b < d: \quad \Lambda_a + \Lambda_b \quad \subseteq \quad \Lambda_{a+b} \qquad (\mathrm{GMI}_\Lambda)$$

Proof First, we prove ($\mathrm{GMI}_\oplus$) $\Rightarrow$ (GMI_Λ). Denote $a = x \oplus y$ and $b = u \oplus v$. Observe that (GMI_Λ) is apparently satisfied if $a + b \geq d$ since, in such a case, $\Lambda_{a+b} = [0, \infty]^2$. Assume therefore $a + b < d$ and assume a, b to be fixed and $x, y, u, v \in [0, \infty]$ to be variable. The left-hand side of ($\mathrm{GMI}_\oplus$), which is the value of $\oplus$ in the point $(x + u, y + v)$, is supposed to be less than or equal to the right-hand side, which is equal to $a + b$. This means, due to the monotonicity of $\oplus$, that the point $(x + u, y + v)$ must be located inside the area of Λ_{a+b}; confer with Figure 1. This fact must hold true not only for every $(x, y) \in L_a$ and every

$(u,v) \in L_b$ but, apparently, also for every $(x,y) \in \Lambda_a$ and $(u,v) \in \Lambda_b$. Thus, if we sum every point of Λ_a with every point of Λ_b we have to obtain a subset of Λ_{a+b}; this is what (GMI$_\Lambda$) expresses.

Now, we prove (GMI$_\Lambda$) $\Rightarrow$ (GMI$_\oplus$). Suppose $a, b \in [0,\infty]$ and $x, y, u, v \in [0,\infty]$ such that $a = x \oplus y$ and $b = u \oplus v$. If $a + b \geq d$ then (GMI$_\oplus$) is apparently satisfied since, thanks to the properties of $\oplus$, its left-hand side is always less than or equal to d. Assume therefore $a + b < d$. Equation (GMI$_\Lambda$) states that, for every $(x,y) \in \Lambda_a$ and $(u,v) \in \Lambda_b$, we have

$$\begin{aligned} (x+u, y+v) &\in \Lambda_{a+b}, \\ (x+u) \oplus (y+v) &\leq a+b. \end{aligned}$$

Hence we obtain (GMI$_\oplus$). ■

Let us introduce some more notions and facts related to the level set plot of $\oplus$ and to Minkowski sum. Assume $A, B, C, D \in [0,\infty]^2$ and $\alpha, \beta \in [0,\infty[$. We have that

$$A \subseteq C \quad \text{and} \quad B \subseteq D \quad \text{implies} \quad A + B \subseteq C + D.$$

The scalar multiple of A by α is defined by

$$\alpha A = \{(\alpha x, \alpha y) \mid (x,y) \in A\}.$$

If A is a convex set then

$$(\alpha + \beta)A = \alpha A + \beta A.$$

For $a, b \in [0,\infty]$ and the two corresponding level cuts of $\oplus$, Λ_a and Λ_b respectively, we define

$$\Lambda_a \leq \Lambda_b \quad \text{iff} \quad \frac{1}{a}\Lambda_a \subseteq \frac{1}{b}\Lambda_b.$$

Observe that the latter defined binary relation $\leq$ is both reflexive and transitive. However, it is generally neither anti-symmetric nor symmetric. Hence it is a preorder.

Observation 4.2 *If $a = 0$ or $b = 0$ then $\Lambda_a + \Lambda_b = \Lambda_{a+b}$; hence $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$.*

Lemma 4.3 *If, for some $a, b \in [0,\infty]$, $\Lambda_a \leq \Lambda_{a+b}$ and $\Lambda_b \leq \Lambda_{a+b}$ then $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$.*

Proof From the assumptions we have $\Lambda_a \subseteq \frac{a}{a+b}\Lambda_{a+b}$ and $\Lambda_b \subseteq \frac{b}{a+b}\Lambda_{a+b}$. The proof is finished by summing these two inequalities. ■

5 Sufficient condition

Assume f and $f^{(-1)}$ according to Assumptions 3.4, assume f to be convex on $[0,d]$, and let $\oplus$ be given according to (2).

Lemma 5.1 *If, for some $a,b \in [0,d]$, we have*

$$\forall x \in [0,1]: \quad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)}$$

then $\Lambda_a \leq \Lambda_b$.

Proof Observe that $(x,y) \in \frac{1}{a}\Lambda_a$ resp. $(x,y) \in \frac{1}{b}\Lambda_b$ if, and only if,

$$\begin{aligned} (ax,ay) \in \Lambda_a \quad &\text{resp.} \quad (bx,by) \in \Lambda_b, \\ f(ax)+f(ay) \leq f(a) \quad &\text{resp.} \quad f(bx)+f(by) \leq f(b), \end{aligned}$$

$$\frac{f(ax)}{f(a)} + \frac{f(ay)}{f(a)} \leq 1 \quad \text{resp.} \quad \frac{f(bx)}{f(b)} + \frac{f(by)}{f(b)} \leq 1.$$

Hence $(x,y) \in \frac{1}{a}\Lambda_a$ implies $(x,y) \in \frac{1}{b}\Lambda_b$. ■

The function f is said to be, for a given $k \in [0,d]$, *k-subscalable* on $[0,d]$ if

$$\forall a,b \in [0,d]\,, x \in [0,1]\,, b-a \geq k\colon \quad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)}.$$

For $a,b \in [0,d]$ we define $f_{\downarrow}^{a,b}\colon\ [0,a] \to [0,f(a)]$ by

$$f_{\downarrow}^{a,b}(x) = \frac{f(a)}{f(b)} f\left(\frac{b}{a}x\right).$$

Apparently, $f_{\downarrow}^{a,b}$ is an increasing bijection and f is k-subscalable on $[0,d]$ if, and only if,

$$\forall a,b \in [0,d]\,, x \in [0,a]\,, b-a \geq k\colon \quad f_{\downarrow}^{a,b}(x) \leq f(x).$$

Lemma 5.2 *If f is, for some $k \in [0,d]$, k-subscalable on $[0,d]$ then*

$$\forall a,b \in [0,\infty[\,,\ \ a \geq k,\ \ b \geq k\colon \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$

Proof If $a+b \geq d$ then apparently $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$ since $\Lambda_{a+b} = [0,\infty]^2$. If $a+b < d$ then, by the definition of k-subscalability and by Lemma 5.1, we have $\Lambda_a \leq \Lambda_{a+b}$ and $\Lambda_b \leq \Lambda_{a+b}$. Lemma 4.3 finishes the proof. ■

The function f is said to be, for a given $k \in [0,d]$, *linear on* $[0,k]$ if there is $r \in \,]0,\infty[$ such that $f(x) = rx$ for every $x \in [0,k]$. Note that we admit also the extreme case $k = 0$ when no linearity is required. We denote

$$\Delta_a = \{(x,y) \in [0,\infty]^2 \mid x+y \leq a\}.$$

Clearly, if f is linear on $[0,k]$ then $\Lambda_a = \Delta_a$ for every $a \in [0,k]$.

Lemma 5.3 *For every* $a, b \in [0, \infty[$ *and for every* $(x, y) \in \Lambda_b$ *we have* $(x + a, y) \in \Lambda_{a+b}$ *and* $(x, y + a) \in \Lambda_{a+b}$.

Proof If $a + b \geq d$ then the proof is apparent since $\Lambda_{a+b} = [0, \infty]^2$. Assume $a+b < d$. We prove $(x+a, y) \in \Lambda_{a+b}$; the proof of $(x, y+a) \in \Lambda_{a+b}$ is analogous. From $(x, y) \in \Lambda_b$ we have $f(x) + f(y) \leq f(b)$; hence $x \leq b$. Since f is strictly increasing and convex on $[0, d]$ we have $f(x + a) - f(x) \leq f(b + a) - f(b)$. The proof is concluded by summing these two inequalities. ■

Lemma 5.4 *For every* $a, b \in [0, \infty[$ *we have* $\Delta_a + \Lambda_b \subseteq \Lambda_{a+b}$.

Proof If $a + b \geq d$ then apparently $\Delta_a + \Lambda_b \subseteq \Lambda_{a+b}$ since $\Lambda_{a+b} = [0, \infty]^2$. Assume $a + b < d$. Since f is convex on $[0, d]$, Λ_{a+b} is a convex set. Take any $(x, y) \in \Lambda_b$. Then $\Delta_a + (x, y)$ is a triangle with the vertices (x, y), $(x + a, y)$, and $(x, y + a)$. According to Lemma 5.3, all these vertices are contained in Λ_{a+b} as well as the whole triangle. ■

Lemma 5.5 *If* f *is, for some* $k \in [0, d]$, *linear on* $[0, k]$ *then*

$$\forall a, b \in [0, \infty[\,, \quad a \leq k \ \text{ or } \ b \leq k: \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$

Proof The lemma is a corollary of Lemma 5.4. ■

The new sufficient condition now follows.

Theorem 5.6 *Consider functions* f *and* $f^{(-1)}$, *which comply with Assumptions 3.4, such that* f *is, for some* $k \in [0, d]$,

- *convex on* $[0, d]$,
- k*-subscalable on* $[0, d]$,
- *linear on* $[0, k]$.

Then f *solves the generalized Mulholland inequality.*

Proof The proof is done invoking Lemma 5.2 and Lemma 5.5. ■

The introduced sufficient condition is not weaker than the one presented by Theorem 3.5 as the following proposition states.

Proposition 5.7 *The assumptions of Theorem 3.5 imply the assumptions of Theorem 5.6.*

Proof We are going to show that, under the assumptions of Theorem 3.5, f is both 0-subscalable on $[0, d]$ and linear on the one-point interval $[0, 0]$. The latter statement is apparent, let us focus on the first one.

The function f is geometrically convex on $]0,d[$ if, and only if, the function $F\colon\ ]-\infty,\log d[\ \to\ ]-\infty,\log e[$, $F=\log\circ f\circ\exp$, is convex. Observe that F is an increasing bijection. Therefore, it is convex if, and only if,

$$F(A)-F(A+X)\ \le\ F(B)-F(B+X)$$

for every $X\in\ ]-\infty,0[$ and $A,B\in\ ]-\infty,\log d[$ such that $A\le B$. Using the substitutions $a=\exp A$, $b=\exp B$, and $x=\exp X$ we obtain

$$\forall a,b\in[0,d]\,,\ \ x\in[0,1]\,,\ \ a\le b\colon\quad \frac{f(ax)}{f(a)}\ge\frac{f(bx)}{f(b)}\,.$$

The latter formula represents the 0-subscalability of f. ■

The introduced sufficient condition is even strictly stronger than the one presented by Theorem 3.5. In order to prove this, we are going to present a parametric class of functions which comply with the assumptions of Theorem 5.6 but not with the assumptions of Theorem 3.5.

Example 5.8 *Let $s\in\left[\frac{1}{2},1\right[$ and $r\in\ ]0,s[$ be a pair of parameters. Function $g_{r,s}\colon\ [0,\infty]\to[0,\infty]$ is defined, for $x\in[0,\infty]$, by*

$$g_{r,s}(x)=\begin{cases}\frac{r}{s}x & \text{if } x\in[0,s[\,,\\ \frac{1-r}{1-s}x-\frac{s-r}{1-s} & \text{if } x\in[s,1[\,,\\ 1 & \text{if } x\in[1,\infty]\,.\end{cases}$$

Observe that $g_{r,s}$ is a function convex on $[0,1]$ which complies with Assumptions 3.4 with $d=e=1$.

Lemma 5.9 *The function $g_{r,s}$ is not geometrically convex on $]0,1[$ for any admissible pair (r,s).*

Proof If $g_{r,s}$ is geometrically convex on $]0,1[$ then $G_{r,s}=\log\circ g_{r,s}\circ\exp$ is convex on $]-\infty,0[$. For $x\in\ ]\log s,0[$ we have

$$\begin{aligned}G_{r,s}(x)&=\log\left(\frac{1-r}{1-s}\mathrm{e}^x-\frac{s-r}{1-s}\right),\\ G'_{r,s}(x)&=\frac{\frac{1-r}{1-s}\mathrm{e}^x}{\frac{1-r}{1-s}\mathrm{e}^x-\frac{s-r}{1-s}},\\ G''_{r,s}(x)&=-\frac{\frac{s-r}{1-s}\frac{1-r}{1-s}\mathrm{e}^x}{\left(\frac{1-r}{1-s}\mathrm{e}^x-\frac{s-r}{1-s}\right)^2}.\end{aligned}$$

However, as we can see, $G''_{r,s}<0$ on the whole $]\log s,0[$. ■

Lemma 5.10 *The function $g_{r,s}$ is s-subscalable on $[0,1]$ and linear on $[0,s]$.*

Proof The linearity on $[0, s]$ is apparent. To show the s-subscalability on $[0, 1]$, take $a, b \in [0, 1]$ such that $b - a \geq s$. Observe that, necessarily, $a \in [0, s]$ and $b \in [s, 1]$. We need to show that

$$g_{\downarrow}^{a,b}(x) \quad = \quad \frac{g_{r,s}(a)}{g_{r,s}(b)} g_{r,s}\left(\frac{b}{a}x\right) \quad \leq \quad g_{r,s}(x)$$

holds for every $x \in [0, a]$. This is however true since $g_{\downarrow}^{a,b}(0) = g_{r,s}(0)$, $g_{\downarrow}^{a,b}(a) = g_{r,s}(a)$, $g_{\downarrow}^{a,b}(x)$ is a convex function, and $g_{r,s}(x)$ is a linear function on $[0, a]$. ■

6 Necessary condition

By Theorem 4.1, we have obtained a reformulation of the generalized Mulholland inequality which is quantified by two variables only. This reformulation has, moreover, the following geometric interpretation. Let us take $a, b \in [0, \infty]$ such that $a + b < d$. If we shift the level cut Λ_a such that its bottom-left corner coincides with the level set L_b then this shifted level cut must remain contained in the level cut Λ_{a+b}, i.e., "below" the level set L_{a+b}; see an illustration in Figure 1. If $a + b \geq d$ then nothing is required. This geometric interpretation gives us an inspiration for a necessary condition based on directional derivatives.

Let $\varphi\colon [0, \infty]^2 \to [0, \infty]$ be a function of two variables. By $\frac{\partial \varphi}{\partial \mathbf{w}}(x, y)$ we denote, if it is defined, the directional derivative of φ along a given positive unit vector $\mathbf{w} = (w_1, w_2)$, $w_1, w_2 \in [0, 1]$, $\sqrt{w_1^2 + w_2^2} = 1$, at a given point $(x, y) \in [0, \infty[^2$. Let us recall the definition which is

$$\frac{\partial \varphi}{\partial \mathbf{w}}(x, y) \quad = \quad \lim_{t \to 0_+} \frac{\varphi\big((x, y) + t\mathbf{w}\big) - \varphi(x, y)}{t}.$$

The partial derivative of φ with respect to a variable t we denote by $\frac{\partial \varphi}{\partial t}$.

Lemma 6.1 *Assume f and $f^{(-1)}$ according to Assumptions 3.4 and let $\oplus$ be the pseudo-addition generated by f according to* (2).

If f and $f^{(-1)}$ satisfy the generalized Mulholland inequality then, for every positive unit vector $\mathbf{w} = (w_1, w_2)$, $w_1, w_2 \in [0, 1]$, $\sqrt{w_1^2 + w_2^2} = 1$, and for every point $(x, y) \in [0, \infty[^2$ we have

$$\frac{\partial \oplus}{\partial \mathbf{w}}(0, 0) \;\; \geq \;\; \frac{\partial \oplus}{\partial \mathbf{w}}(x, y)$$

if the corresponding derivatives are defined.

Proof Since $\oplus$ is continuous and increasing in both variables, we have $\frac{\partial \oplus}{\partial \mathbf{w}}(x, y) \geq 0$ in every point $(x, y) \in [0, \infty[^2$ where it is defined. If $(x, y) \notin \operatorname{supp} \oplus$ then

$\frac{\partial\oplus}{\partial\mathbf{w}}(x,y) = 0$ and the conclusion of the lemma is apparent. Suppose $(x,y) \in \operatorname{supp}\oplus$. By the definition of directional derivative we have

$$\frac{\partial\oplus}{\partial\mathbf{w}}(0,0) = \lim_{t\to 0_+} \frac{(0+t\,w_1)\oplus(0+t\,w_2)-(0\oplus 0)}{t}, \tag{3}$$

$$\frac{\partial\oplus}{\partial\mathbf{w}}(x,y) = \lim_{t\to 0_+} \frac{(x+t\,w_1)\oplus(y+t\,w_2)-(x\oplus y)}{t}. \tag{4}$$

Since $0\oplus 0 = 0$, (3) can be simplified to

$$\frac{\partial\oplus}{\partial\mathbf{w}}(0,0) = \lim_{t\to 0_+} \frac{t\,w_1\oplus t\,w_2}{t} \tag{5}$$

which is, actually, the right derivative of the expression $t\,w_1\oplus t\,w_2$ according to t in zero:

$$\frac{\partial\oplus}{\partial\mathbf{w}}(0,0) = \frac{\partial}{\partial t}\Big(t\,w_1\oplus t\,w_2\Big)_{t=0_+}. \tag{6}$$

Similarly, the right hand part of (4) is the right derivative of the expression $(x+t\,w_1)\oplus(y+t\,w_2)-(x\oplus y)$ according to t in zero. Since, in such a case, $x\oplus y$ is considered as a constant, we obtain:

$$\begin{aligned}\frac{\partial\oplus}{\partial\mathbf{w}}(x,y) &= \frac{\partial}{\partial t}\Big[(x+t\,w_1)\oplus(y+t\,w_2)-(x\oplus y)\Big]_{t=0_+}\\ &= \frac{\partial}{\partial t}\Big[(x+t\,w_1)\oplus(y+t\,w_2)\Big]_{t=0_+}. \end{aligned}\tag{7}$$

By (GMI$_\oplus$) we have, for every $t\geq 0$,

$$(x\oplus y)+(t\,w_1\oplus t\,w_2) \geq (x+t\,w_1)\oplus(y+t\,w_2).$$

Observe that both sides of the inequality are equal when $t=0$. Therefore, their right partial derivatives with respect to t at zero must satisfy the same:

$$\frac{\partial}{\partial t}\Big[(x\oplus y)+(t\,w_1\oplus t\,w_2)\Big]_{t=0_+} \geq \frac{\partial}{\partial t}\Big[(x+t\,w_1)\oplus(y+t\,w_2)\Big]_{t=0_+}.$$

As $x\oplus y$ in the left-hand side derivative plays the role of a constant, we obtain that (6) is greater than or equal to (7) which finishes the proof. ■

7 Compositions of solutions of the generalized Mulholland inequality

In this section we are going to show that the set of functions that solve the generalized Mulholland inequality is not closed with respect to compositions. This fact has been, actually, aready proven by a previous paper [13] in which case, however, the counter-example has been based on functions that were bijections of $[0,\infty]$. Here we are going to show that a similar counter-example can be made also on those solutions of the generalized Mulholland inequality that are *not* bijections of $[0,\infty]$. As a corollary of this fact, the next section will state that the dominance relation is *not* transitive on the set of nilpotent t-norms.

Observation 7.1 *It can be easily checked that the function* $h\colon [0,\infty]\to[0,\infty]$*, defined for* $p>1$ *by*

$$h(x) \quad = \quad \begin{cases} x^p & \textit{if } x\in[0,1]\,, \\ 1 & \textit{if } x>1, \end{cases}$$

is both convex and geometrically convex on $]0,1[$*. Hence by Theorem 3.5 it satisfies the generalized Mulholland inequality.*

Lemma 7.2 *Assume, for some fixed* $s\in\left]\frac{1}{2},1\right[$ *and* $r\in]0,s[$*, the function* $g_{r,s}$ *from Example 5.8; denote it just by* g*. Assume further, for some fixed* $p>1$*, the function* h *from Observation 7.1. While both* g *and* h *satisfy the generalized Mulholland inequality,* $f=g\circ h$ *does not.*

Proof Let $\oplus$ be the pseudo-addition generated by f according to (2). We are going to show that it does not satisfy ($\mathrm{GMI}_\oplus$) by showing that the conclusion of Lemma 6.1 is violated.

Observe that we have, for $x,y\in[0,\infty]$,

$$f(x) \quad = \quad \begin{cases} \frac{r}{s}x^p & \text{if } x\in\left[0,s^{\frac{1}{p}}\right[, \\ \frac{1-r}{1-s}x^p-\frac{s-r}{1-s} & \text{if } x\in\left[s^{\frac{1}{p}},1\right[, \\ 1 & \text{if } x\geq 1, \end{cases}$$

$$f^{(-1)}(x) \quad = \quad \begin{cases} \left(\frac{s}{r}x\right)^{\frac{1}{p}} & \text{if } x\in[0,r[\,, \\ \left(\frac{1-s}{1-r}x+\frac{s-r}{1-r}\right)^{\frac{1}{p}} & \text{if } x\in[r,1[\,, \\ 1 & \text{if } x\geq 1, \end{cases}$$

$$x\oplus y \quad = \quad \begin{cases} (x^p+y^p)^{\frac{1}{p}} & \text{if } x,y\geq 0 \text{ and } x^p+y^p<s, \\ \left(x^p+y^p-\frac{s-r}{1-r}\right)^{\frac{1}{p}} & \text{if } x,y\geq s^{\frac{1}{p}} \text{ and } \left(x^p+y^p-\frac{s-r}{1-r}\right)^{\frac{1}{p}}<1, \\ \dots & \text{(we omit the rest).} \end{cases}$$

Let us take the unit vector $\mathbf{w}=(1/\sqrt{2},1/\sqrt{2})$ and observe that $\oplus$ is a smooth function when restricted to $\left\{(x,y)\in[0,\infty]^2 \mid x^p+y^p<s\right\}$. Therefore the directional derivative of $\oplus$ along $\mathbf{w}$ at $(0,0)$ exists and is given by (confer

with (6)):

$$\begin{aligned}\frac{\partial\oplus}{\partial\mathbf{w}}(0,0) &= \frac{\partial}{\partial t}\left[\frac{t}{\sqrt{2}}\oplus\frac{t}{\sqrt{2}}\right]_{t=0_+}\\ &= \frac{\partial}{\partial t}\left[\left(\left(\frac{t}{\sqrt{2}}\right)^p+\left(\frac{t}{\sqrt{2}}\right)^p\right)^{\frac{1}{p}}\right]_{t=0_+} = \frac{2^{\frac{1}{p}}}{\sqrt{2}}.\end{aligned}$$

Observe that the area $\left\{(x,y)\in\left[s^{\frac{1}{p}},\infty\right]^2 \mid \left(x^p+y^p-\frac{s-r}{1-r}\right)^{\frac{1}{p}}<1\right\}$ in the above definition of $\oplus$ is non-empty. Indeed, since $s<1$ and $r<\frac{1}{2}$ we can derive

$$\begin{aligned}2r &< 1,\\ 2r(1-s) &< 1-s,\\ s+r-2rs &< 1-r,\\ 2s-2rs+r-s &< 1-r,\\ 2s-\frac{s-r}{1-r} &< 1,\\ \left(\left(s^{\frac{1}{p}}\right)^p+\left(s^{\frac{1}{p}}\right)^p-\frac{s-r}{1-r}\right)^{\frac{1}{p}} &< 1.\end{aligned}$$

Furthermore, when restricted to this area, $\oplus$ is obviously a smooth function.

Therefore the directional derivative of $\oplus$ along $\mathbf{w}$ at $(s^{\frac{1}{p}},s^{\frac{1}{p}})$ exists, as well, and is given by (confer with (7)):

$$\begin{aligned}\frac{\partial\oplus}{\partial\mathbf{w}}\left(s^{\frac{1}{p}},s^{\frac{1}{p}}\right) &= \frac{\partial}{\partial t}\left[\left(s^{\frac{1}{p}}+\frac{t}{\sqrt{2}}\right)\oplus\left(s^{\frac{1}{p}}+\frac{t}{\sqrt{2}}\right)\right]_{t=0_+}\\ &= \frac{\partial}{\partial t}\left[\left(\left(s^{\frac{1}{p}}+\frac{t}{\sqrt{2}}\right)^p+\left(s^{\frac{1}{p}}+\frac{t}{\sqrt{2}}\right)^p-\frac{s-r}{1-r}\right)^{\frac{1}{p}}\right]_{t=0_+}\\ &= \left[\frac{2}{\sqrt{2}}\left(2\left(s^{\frac{1}{p}}+\frac{t}{\sqrt{2}}\right)^p-\frac{s-r}{1-r}\right)^{\frac{1-p}{p}}\left(s^{\frac{1}{p}}+\frac{t}{\sqrt{2}}\right)^{p-1}\right]_{t=0_+}\\ &= \frac{2}{\sqrt{2}}\left(2s-\frac{s-r}{1-r}\right)^{\frac{1-p}{p}}s^{\frac{p-1}{p}} = \frac{2}{\sqrt{2}}\left(1+\frac{r}{s}\cdot\frac{1-s}{1-r}\right)^{\frac{1-p}{p}}.\end{aligned}$$

Since the expression $\frac{r}{s}\cdot\frac{1-s}{1-r}$ is always strictly greater than zero and since $p>1$, we have

$$\frac{\partial\oplus}{\partial\mathbf{w}}(0,0) < \frac{\partial\oplus}{\partial\mathbf{w}}\left(s^{\frac{1}{p}},s^{\frac{1}{p}}\right).$$

This however, together with Lemma 6.1, means that $\oplus$ does not satisfy (GMI$_\oplus$); hence f does not satisfy the generalized Mulholland inequality. ■

8 Transitivity of dominance on nilpotent t-norms

The result of Lemma 7.2 gives us the following corollary.

Theorem 8.1 *The relation of dominance is not transitive on the set of nilpotent t-norms.*

Proof Define functions g, h, and $g \circ h$ as in the proof of Lemma 7.2. Let t_2 be the additive generator of any nilpotent t-norm such that $t_2(0) = 1$; denote the t-norm generated by t_2 as $*_2$. Define $t_1 = g \circ t_2$ and $t_3 = h^{(-1)} \circ t_2$ and observe that both t_1 and t_3 are generators of nilpotent t-norms; denote these t-norms by $*_1$ and $*_3$, respectively. We have $t_1 \circ t_2^{(-1)} = g$, hence $*_1$ dominates $*_2$, and we have $t_2 \circ t_3^{(-1)} = h$, hence $*_2$ dominates $*_3$. However, since $t_1 \circ t_3^{(-1)} = g \circ h$, which by Lemma 7.2 does not solve the generalized Mulholland inequality, $*_1$ does not dominate $*_3$. ■

Thus we can construct a counter-example of three nilpotent t-norms that violate the transitivity of the dominance relation. The recent result [14] has shown that we can do the same for strict t-norms. Hence we can claim that the dominance relation is generally not transitive on continuous Archimedean t-norms.

Acknowledgments

This work was supported by the Czech Science Foundation under Project GJ15-07724Y. The author would like to thank the anonymous referee for his comments and Přemysl Jedlička and Jan Hora for their help.

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