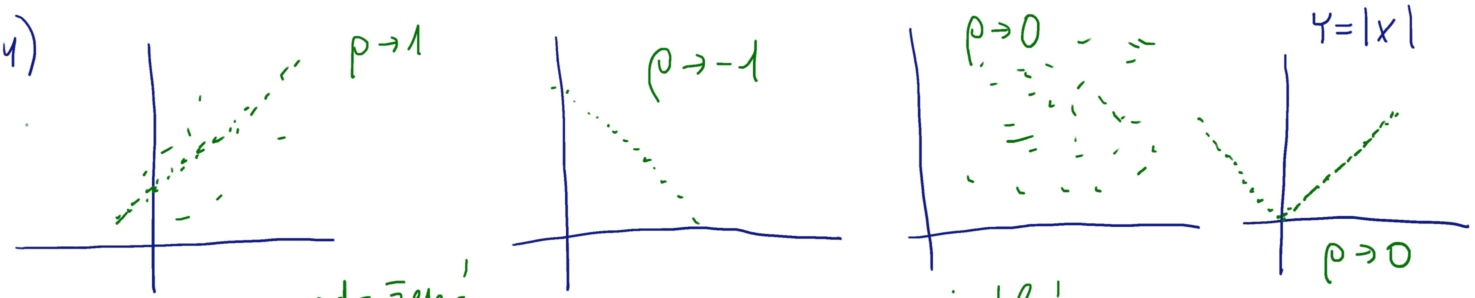
(X,Y)



④ (X,Y)

sdružená — marginální

$x \backslash y$	0	1	$P[X=x]$
2	$1/6$	$1/12$	$3/12 = P[X=2]$
4	$1/2$	$1/4$	$3/4 = P[X=4]$
$P[Y=y]$	$4/6$	$4/12$	$12/12 = 1$

$$EX = 2\cdot\frac{3}{12} + 4\cdot\frac{3}{4} = \frac{1}{2} + 3 = 3{,}5$$

$$E(X^2) = 4\cdot\frac{3}{12} + 16\cdot\frac{3}{4} = 1 + 12 = 13$$

$$DX = 13 - 3{,}5^2 = \frac{3}{4}$$

$$EY = 0\cdot\frac{4}{6} + 1\cdot\frac{4}{12} = \frac{1}{3}$$

$$E(Y^2) = 0^2\frac{4}{6} + 1^2\frac{4}{12} = \frac{1}{3}$$

$$DY = \frac{1}{3} - \frac{1}{3^2} = \frac{2}{9}$$

$$\operatorname{cov}(X,Y) = E(X\cdot Y) - EX\cdot EY$$

$$E(X\cdot Y) = \sum_{x,y} x\cdot y\cdot P[X=x, Y=y]$$

$$= 2\cdot 0\cdot\frac{1}{6} + 2\cdot 1\cdot\frac{1}{12} + 4\cdot 0\cdot\frac{1}{2} + 4\cdot 1\cdot\frac{1}{4} = \frac{1}{6} + 1 = \frac{7}{6}$$

X,Y nezávislé $\Leftrightarrow$

$\Leftrightarrow \forall x,y: P[X=x]\cdot P[Y=y] = P[X=x, Y=y]$

$$\operatorname{cov}(X,Y) = \frac{7}{6} - 3{,}5\cdot\frac{1}{3} = \frac{7}{6} - \frac{7}{2}\cdot\frac{1}{3} = 0$$

$$\frac{3}{12}\cdot\frac{4}{6} \stackrel{?}{=} \frac{1}{6} \qquad \frac{3}{12}\cdot\frac{4}{12} \stackrel{?}{=} \frac{1}{12}$$

$$\frac{3}{4}\cdot\frac{4}{6} \stackrel{?}{=} \frac{1}{2} \qquad \frac{3}{4}\cdot\frac{4}{12} \stackrel{?}{=} \frac{1}{4}$$

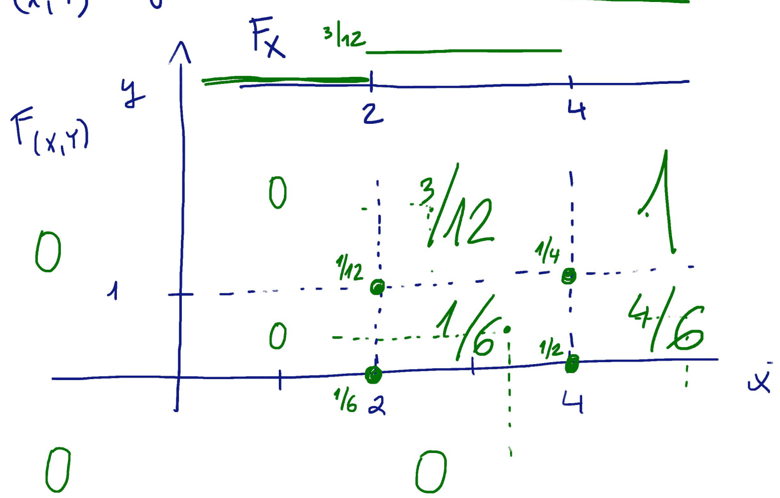
$$\rho(X,Y) = 0$$

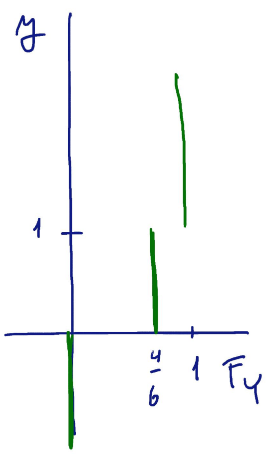
ano, X,Y jsou nezávislé

$x \backslash y$	0	1
2	$1/6$	$1/12$
4	$1/2$	$1/4$

$$F_{(X,Y)}(x,y) = P[X\le x, Y\le y],$$

$$F_Y(y) = \lim_{x\to\infty} F_{(X,Y)}(x,y)$$





⑤ (X,Y)

$x \backslash y$	0	1	2	$P[X=x]$
0	1/8	0	1/8	2/8
1	1/8	1/8	1/8	3/8
3	0	3/8	0	3/8
$P[Y=y]$	2/8	4/8	2/8	1

$$EX = 0\cdot 2/8 + 1\cdot 3/8 + 3\cdot 3/8 = \frac{3+9}{8} = \frac{12}{8} = \frac{3}{2}$$

$$E(X^2) = 0\cdot\frac{2}{8} + 1\cdot\frac{3}{8} + 9\cdot\frac{3}{8} = \frac{3+27}{8} = \frac{30}{8} = \frac{15}{4}$$

$$DX = \frac{15}{4} - \frac{9}{4} = \frac{6}{4} = \frac{3}{2}$$

$$\mathrm{cov}(X,Y) = E(X\cdot Y) - EX\cdot EY$$

$$E(X\cdot Y) = \sum_{x,y} x\cdot y\cdot P[X=x, Y=y]$$

$$= 0\cdot 0\cdot\frac{1}{8} + 0\cdot 1\cdot 0 + 0\cdot 2\cdot\frac{1}{8}$$

$$+ 1\cdot 0\cdot\frac{1}{8} + 1\cdot 1\cdot\frac{1}{8} + 1\cdot 2\cdot\frac{1}{8}$$

$$+ 3\cdot 0\cdot 0 + 3\cdot 1\cdot\frac{3}{8} + 3\cdot 2\cdot 0$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{9}{8} = \frac{12}{8} = \frac{3}{2}$$

$$EY = 0\cdot\frac{2}{8} + 1\cdot\frac{4}{8} + 2\cdot\frac{2}{8} = \frac{4+4}{8} = 1$$

$$E(Y^2) = 0\cdot\frac{2}{8} + 1\cdot\frac{4}{8} + 4\cdot\frac{2}{8} = \frac{4+8}{8} = \frac{12}{8} = \frac{3}{2}$$

$$DY = \frac{3}{2} - 1 = \frac{1}{2}$$

$$\mathrm{cov}(X,Y) = \frac{3}{2} - \frac{3}{2}\cdot 1 = 0$$

$$\Rightarrow \rho(X,Y) = 0$$

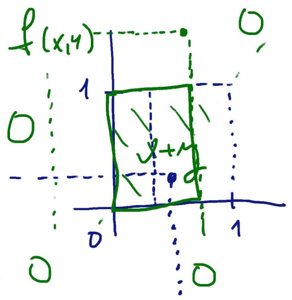
jsou X,Y nezávislé ?

$$\underbrace{\underset{=2/8}{P[X=0]} \cdot \underset{=2/8}{P[Y=0]}}_{\frac{4}{16} = \frac{1}{4}} \overset{?}{=} \underset{=1/8}{P[X=0, Y=0]}$$

$$\frac{4}{16} = \frac{1}{4} \neq \frac{1}{8} \text{ !} \Rightarrow X,Y \text{ nejsou nezávislé !}$$

(10) (X,Y) sdružená hustota $f_{(X,Y)}(x,y) = \begin{cases} x+y & \text{pokud } (x,y) \in \langle 0,1\rangle^2 \\ 0 & \text{jinak} \end{cases}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dy$$



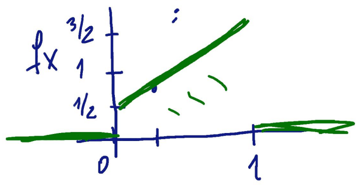
$x \in \langle 0,1\rangle$:

$$f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$$

$y \in \langle 0,1\rangle$:

$$f_Y(y) = y + \frac{1}{2}$$

$$f_Y(x) = f_X(x) = \begin{cases} x + 1/2 & \dots x \in \langle 0,1\rangle \\ 0 & \dots \text{jinak} \end{cases}$$

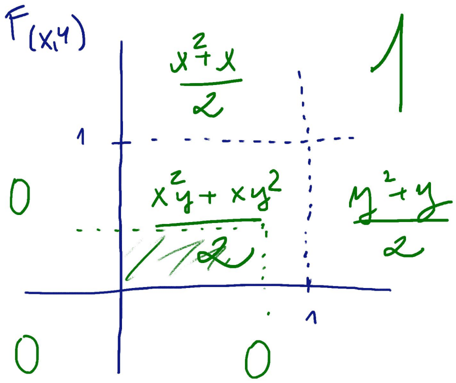


jsou X,Y nezávislé?

$$f_{(X,Y)}(x,y) \overset{?}{=} f_X(x) \cdot f_Y(y)$$

$(x,y) \in \langle 0,1\rangle^2$: $\quad x+y \overset{?}{=} (x+\frac{1}{2})(y+\frac{1}{2}) \quad \dots$ NE

$$F_{(X,Y)}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{(X,Y)}(u,v)\,dv\,du$$



$(x,y) \in \langle 0,1\rangle^2$:

$$F_{(X,Y)}(x,y) = \int_0^x\int_0^y (u+v)\,dv\,du = \int_0^x \left[uv + \frac{v^2}{2}\right]_0^y du = \int_0^x \left(uy + \frac{y^2}{2}\right) du$$

$$= \left[\frac{u^2 y}{2} + \frac{u y^2}{2}\right]_0^x = \frac{x^2 y + x y^2}{2}$$

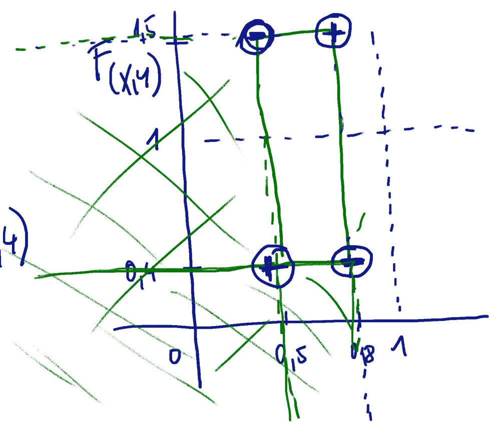
$(x,y) \in \langle 0,1) \times \langle 1,\infty)$:

$$F_{(X,Y)}(x,y) = \int_0^x\int_0^1 (u+v)\,dv\,du = \int_0^x \left[uv + \frac{v^2}{2}\right]_0^1 du = \int_0^x \left(u + \frac{1}{2}\right) du$$

$$= \left[\frac{u^2}{2} + \frac{u}{2}\right]_0^x = \frac{x^2+x}{2}$$

$$P[0{,}5 \le X \le 0{,}8,\ 0{,}4 \le Y \le 1{,}5] =$$

$$F_{(X,Y)}(0{,}8;1{,}5) - F_{(X,Y)}(0{,}5;1{,}5) - F_{(X,Y)}(0{,}8;0{,}4) + F_{(X,Y)}(0{,}5;0{,}4)$$



$$f_{(X,Y)}(x,y) = \begin{cases} x+y & \langle 0,1\rangle^2 \\ 0 & \text{jinde} \end{cases} \qquad f_Y(x) = f_X(x) = \begin{cases} x+\frac{1}{2} & \langle 0,1\rangle \\ 0 & \text{jinde} \end{cases}$$

$$EX = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 \underbrace{x\left(x+\frac{1}{2}\right)}_{x^2+\frac{x}{2}} dx = \left[\frac{x^3}{3}+\frac{x^2}{4}\right]_0^1 = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

$$EY = \frac{7}{12}$$

$$E(X\cdot Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot y\cdot f_{(x,y)}(x,y)\,dy\,dx = \int_0^1 \underbrace{\int_0^1 \overbrace{xy(x+y)}^{x^2y+xy^2}\,dy}\,dx$$

$$\int_0^1 \left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_0^1 dx = \int_0^1 \left(\frac{x^2}{2}+\frac{x}{3}\right)dx = \left[\frac{x^3}{6}+\frac{x^2}{6}\right]_0^1 = \frac{2}{6} = \frac{1}{3}$$

$$\operatorname{cov}(X,Y) = E(X\cdot Y) - EX\cdot EY = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = \frac{48}{144} - \frac{49}{144} = -\frac{1}{144}$$

$\Rightarrow$ X, Y nejsou nezávislé

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 \underbrace{x^2\left(x+\frac{1}{2}\right)}_{x^3+\frac{x^2}{2}} dx = \left[\frac{x^4}{4}+\frac{x^3}{6}\right]_0^1 = \frac{3+2}{12} = \frac{5}{12}$$

$$DX = \frac{5}{12} - \frac{49}{144} = \frac{60-49}{144} = \frac{11}{144} = DY$$

kovarianční matice
$$\begin{bmatrix} \overset{=DX}{\operatorname{cov}(X,X)} & \operatorname{cov}(X,Y) \\ \underset{=\operatorname{cov}(X,Y)}{\operatorname{cov}(Y,X)} & \underset{=DY}{\operatorname{cov}(Y,Y)} \end{bmatrix} = \begin{bmatrix} \frac{11}{144} & -\frac{1}{144} \\ -\frac{1}{144} & \frac{11}{144} \end{bmatrix}$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{DX}\sqrt{DY}} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{-1/144}{11/144} = -\frac{1}{11}$$

korelační mat.
$$\begin{bmatrix} \overset{=1}{\rho(X,X)} & \rho(X,Y) \\ \rho(Y,X) & \underset{=1}{\rho(Y,Y)} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{11} \\ -\frac{1}{11} & 1 \end{bmatrix}$$

(2) (X,Y)

$DX = 9$

$DY = 4$

$\mathrm{cov}(X,Y) = -3$

$U = 2X + 3Y - 4$

$V = 4X - 5Y + 1$

$\mathrm{cov}(X,X) = E(X \cdot X) - EX \cdot EX$

$= E(X^2) - (EX)^2 = DX$

$$\mathrm{cov}(U,V) = \mathrm{cov}(2X+3Y-4,\ 4X-5Y+1)$$
$$= \mathrm{cov}(2X+3Y,\ 4X-5Y)$$
$$= \mathrm{cov}(2X,\ 4X-5Y) + \mathrm{cov}(3Y,\ 4X-5Y)$$
$$= \mathrm{cov}(2X,4X) + \mathrm{cov}(2X,-5Y) + \mathrm{cov}(3Y,4X) + \mathrm{cov}(3Y,-5Y)$$
$$= 8\underbrace{\mathrm{cov}(X,X)}_{DX} - 10\,\mathrm{cov}(X,Y) + 12\,\mathrm{cov}(Y,X) - 15\,\underbrace{\mathrm{cov}(Y,Y)}_{DY}$$
$$= 8 \cdot 9 - 10 \cdot (-3) + 12 \cdot (-3) - 15 \cdot 4 = 6$$

$$DU = D(2X+3Y-4) = D(2X+3Y)$$
$$= D(2X) + D(3Y) + 2\,\mathrm{cov}(2X,3Y)$$
$$= 4DX + 9DY + 2 \cdot 2 \cdot 3\,\mathrm{cov}(X,Y)$$
$$= \ldots$$