(4) (X,Y) $\quad P_{(X,Y)}(x,y) = P[X=x, Y=y]$

sdružené

$x \backslash y$	0	1	$P[X=x]$ ← marginální
2	1/6	1/12	3/12 $= P[X=2]$
4	1/2	1/4	3/4 $= P[X=4]$
$P[Y=y]$	4/6	4/12	$\frac{3}{12}+\frac{9}{12}=\frac{12}{12}=1$

$$EX = 2\cdot\frac{3}{12} + 4\cdot\frac{3}{4} = \frac{6}{12}+3 = 3{,}5 = \frac{7}{2}$$

$$E(X^2) = 2^2\cdot\frac{3}{12} + 4^2\cdot\frac{3}{4} = 1 + 12 = 13$$

$$DX = E(X^2) - EX^2 = 13 - \frac{49}{4} = \frac{52-49}{4} = \frac{3}{4}$$

$$E(X,Y) = (EX, EY) = \left(\frac{7}{2}, \frac{1}{3}\right)$$

$$D(X,Y) = (DX, DY) = \left(\frac{3}{4}, \frac{2}{9}\right)$$

$$EY = 0\cdot\frac{4}{6} + 1\cdot\frac{4}{12} = \frac{4}{12} = \frac{1}{3}$$

$$E(Y^2) = 0^2\cdot\frac{4}{6} + 1^2\cdot\frac{4}{12} = \frac{1}{3}$$

$$DY = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

$$E(X\cdot Y) = 2\cdot 0\cdot\frac{1}{6} + \underline{2\cdot 1\cdot\frac{1}{12}} + 4\cdot 0\cdot\frac{1}{2} + \underline{4\cdot 1\cdot\frac{1}{4}} = \frac{1}{6} + 1 = \frac{7}{6}$$

jsou X,Y nezávislé ? ... ANO

$$\forall x,y: P[X=x, Y=y] \overset{?}{=} P[X=x]\cdot P[Y=y]$$

$$P[X=4, Y=1] \overset{?}{=} P[X=4]\cdot P[Y=1]$$

$$1/4 \overset{?}{=} 3/4 \cdot 4/12 \quad \dots \text{ ANO}$$

$$\operatorname{cov}(X,Y) = \frac{7}{6} - \frac{7}{2}\cdot\frac{1}{3} = 0$$

$$\rho(X,Y) = 0$$

$$\boxed{X,Y \text{ nez.} \Rightarrow \rho(X,Y)=0}$$

(2) (X,Y) $\quad \cancel{EX}$ $\quad D(X,Y) = (9,4)$ $\quad \operatorname{cov}(X,Y) = -3$

$$U = 2X+3Y-4 \qquad V = 4X-5Y+1$$

$$\operatorname{cov}(U,V) = \operatorname{cov}(2X+3Y-4,\ 4X-5Y+1)$$

$$= \operatorname{cov}(2X+3Y,\ 4X-5Y)$$

$$= \operatorname{cov}(2X, 4X-5Y) + \operatorname{cov}(3Y, 4X-5Y)$$

$$= \operatorname{cov}(2X,4X) + \operatorname{cov}(2X,-5Y) + \operatorname{cov}(3Y,4X) + \operatorname{cov}(3Y,-5Y)$$

$$= 8\underbrace{\operatorname{cov}(X,X)}_{DX} - 10\operatorname{cov}(X,Y) + 12\operatorname{cov}(Y,X) - 15\underbrace{\operatorname{cov}(Y,Y)}_{DY}$$

$$= 8\cdot 9 - 10\cdot(-3) + 12\cdot(-3) - 15\cdot 4$$

$$= \dots$$

5 (X,Y)

$x \backslash y$	0	1	2	$P[X=x]$
0	$1/8$	0	$1/8$	$2/8$
1	$1/8$	$1/8$	$1/8$	$3/8$
3	0	$3/8$	0	$3/8$
$P[Y=y]$	$2/8$	$4/8$	$2/8$	$8/8 = 1$

$EX = 0 \cdot \frac{2}{8} + 1 \cdot \frac{3}{8} + 3 \cdot \frac{3}{8} = \frac{3+9}{8} = \frac{12}{8} = \frac{3}{2}$

$E(X^2) = 0^2 \cdot \frac{2}{8} + 1^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{3}{8} = \frac{3+27}{8} = \frac{30}{8} = \frac{15}{4}$

$DX = \frac{15}{4} - \frac{9}{4} = \frac{6}{4} = \frac{3}{2}$

$E(X,Y) = \left(\frac{3}{2}, 1\right)$

$D(X,Y) = \left(\frac{3}{2}, \frac{1}{2}\right)$

$EY = 0 \cdot \frac{2}{8} + 1 \cdot \frac{4}{8} + 2 \frac{2}{8} = \frac{4+4}{8} = 1$

$E(Y^2) = 0^2 \cdot \frac{2}{8} + 1^2 \frac{4}{8} + 2^2 \frac{2}{8} = \frac{4+8}{8} = \frac{12}{8} = \frac{3}{2}$

$DY = \frac{3}{2} - 1^2 = \frac{1}{2}$

$E(X \cdot Y) = 0 \cdot 0 \cdot \frac{1}{8} + 0 \cdot 1 \cdot 0 + 0 \cdot 2 \frac{1}{8}$

$+ 1 \cdot 0 \cdot \frac{1}{8} + \underline{1 \cdot 1 \cdot \frac{1}{8}} + \underline{1 \cdot 2 \frac{1}{8}}$

$+ 3 \cdot 0 \cdot 0 + \underline{3 \cdot 1 \cdot \frac{3}{8}} + 3 \cdot 2 \cdot 0$

$= \frac{1}{8} + \frac{2}{8} + \frac{9}{8} = \frac{12}{8} = \frac{3}{2}$

$\mathrm{cov}(X,Y) = \frac{3}{2} - \frac{3}{2} \cdot 1 = 0$

$\Rightarrow \rho(X,Y) = 0$

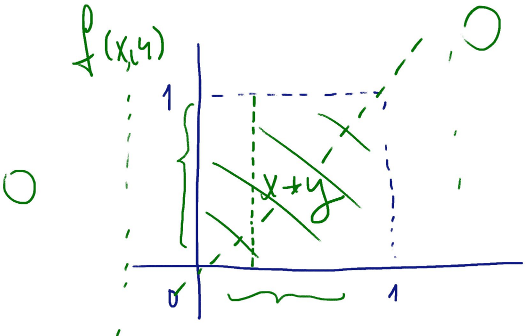
jsou X,Y nezávislé?

$P[X=0] \cdot P[Y=0] \stackrel{?}{=} P[X=0, Y=0]$

$\underbrace{2/8 \cdot 2/8}_{4/16 = 1/4} \stackrel{?}{=} 1/8$

(10) (X,Y) sdružená hustota $f_{(X,Y)}(x,y) = \begin{cases} x+y & \text{pokud } (x,y)\in\langle 0,1\rangle^2 \\ 0 & \text{jinak} \end{cases}$

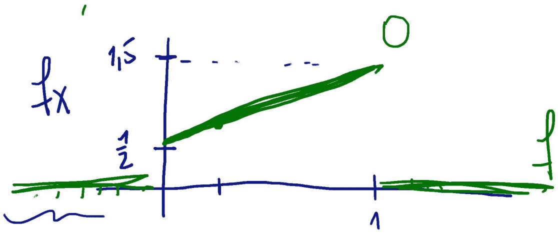
$$\int_0^1\int_0^1 f_{(X,Y)}(x,y)\,dy\,dx = 1$$



$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dy$$

pro $x\in\langle 0,1\rangle$:

$$f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x+\frac{1}{2}$$



$$f_Y(x) = f_X(x) = \begin{cases} x+1/2 & \langle 0,1\rangle \\ 0 & \text{jinde} \end{cases}$$

jsou X,Y nezávislé? $\Leftrightarrow$ $\forall x,y$: $f_{(X,Y)}(x,y) = f_X(x)\cdot f_Y(y)$

$(x,y)\in\langle 0,1\rangle^2$ $\quad x+y \neq (x+\frac{1}{2})(y+\frac{1}{2})$

... NE

$$EY = EX = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 \underbrace{x\left(x+\tfrac{1}{2}\right)}_{x^2+\frac{x}{2}}dx = \left[\frac{x^3}{3}+\frac{x^2}{4}\right]_0^1 = \frac{1}{3}+\frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

$$E(X,Y) = \left(\frac{7}{12}, \frac{7}{12}\right)$$

$$E(X^2) = E(Y^2) = \int_0^1 \underbrace{x^2\left(x+\tfrac{1}{2}\right)}_{x^3+\frac{x^2}{2}}dx = \left[\frac{x^4}{4}+\frac{x^3}{6}\right]_0^1 = \frac{1}{4}+\frac{1}{6} = \frac{3+2}{12} = \frac{5}{12}$$

$$DX = DY = \frac{5}{12} - \frac{49}{144} = \frac{60-49}{144} = \frac{11}{144} \qquad D(X,Y) = \left(\frac{11}{144}, \frac{11}{144}\right)$$

$$E(X\cdot Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot y\cdot f_{(X,Y)}(x,y)\,dy\,dx = \int_0^1\int_0^1 x\cdot y\cdot \overbrace{(x+y)}^{x^2y+xy^2}\,dy\,dx = \int_0^1 \left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_0^1 dx$$

$$= \int_0^1 \left(\frac{x^2}{2}+\frac{x}{3}\right)dx = \left[\frac{x^3}{6}+\frac{x^2}{6}\right]_0^1 = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

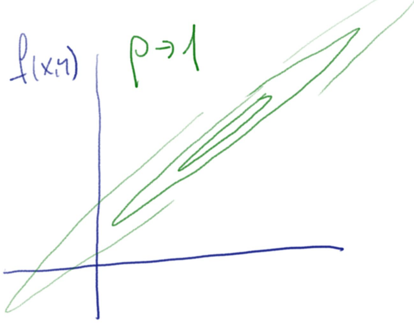
$$\operatorname{cov}(X,Y) = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = \frac{48-49}{144} = -\frac{1}{144}$$ ~~kov. mat.~~

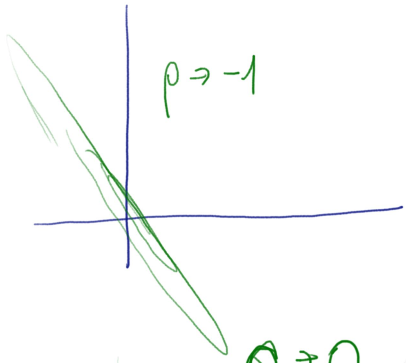
kovarianční matice

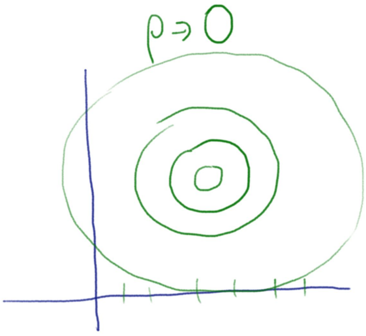
$$\begin{bmatrix} \underbrace{\operatorname{cov}(X,X)}_{=DX} & \operatorname{cov}(X,Y) \\ \underbrace{\operatorname{cov}(Y,X)}_{=\operatorname{cov}(X,Y)} & \underbrace{\operatorname{cov}(Y,Y)}_{=DY} \end{bmatrix} = \begin{bmatrix} 11/144 & -1/144 \\ -1/144 & 11/144 \end{bmatrix}$$

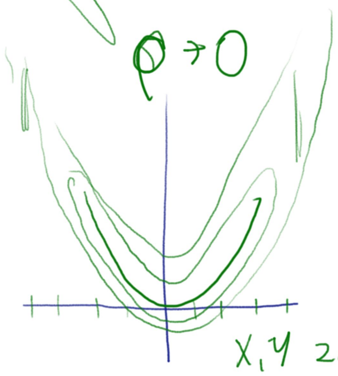
$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{DX}\sqrt{DY}} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{-\frac{1}{144}}{\frac{11}{144}} = -\frac{1}{11}$$

korelační matice ... $\begin{bmatrix} 1 & -\frac{1}{11} \\ -\frac{1}{11} & 1 \end{bmatrix}$

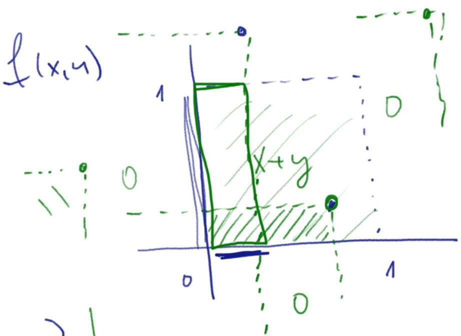






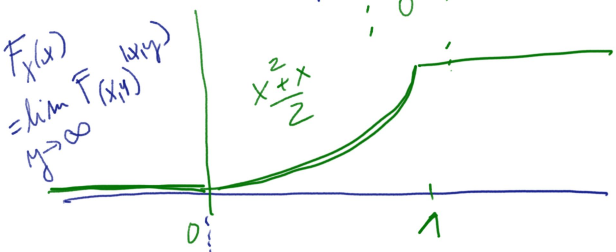


X, Y závislé!



$$F_{(X,Y)}(x,y) = P[X \le x,\ Y \le y]$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{(X,Y)}(u,v)\, dv\, du$$

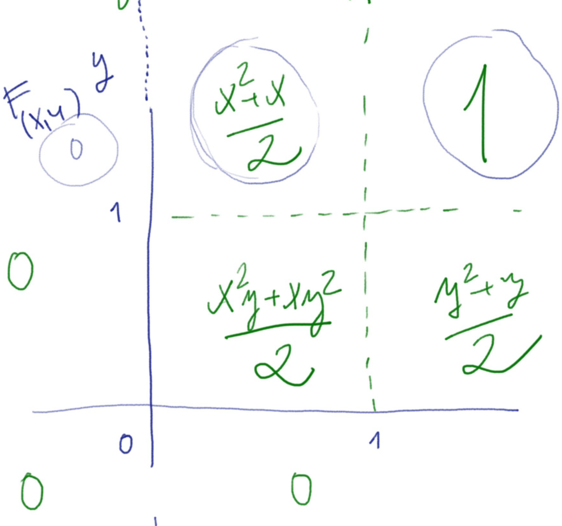


$(x,y) \in (0,1)^2$:

$$F_{(X,Y)}(x,y) = \int_0^x \underbrace{\int_0^y (u+v)\, dv}\, du$$

$$= \int_0^x \left[uv + \frac{v^2}{2}\right]_0^y du = \int_0^x \left(uy + \frac{y^2}{2}\right) du$$

$$= \left[\frac{u^2 y}{2} + \frac{uy^2}{2}\right]_0^x = \frac{x^2 y + x y^2}{2}$$



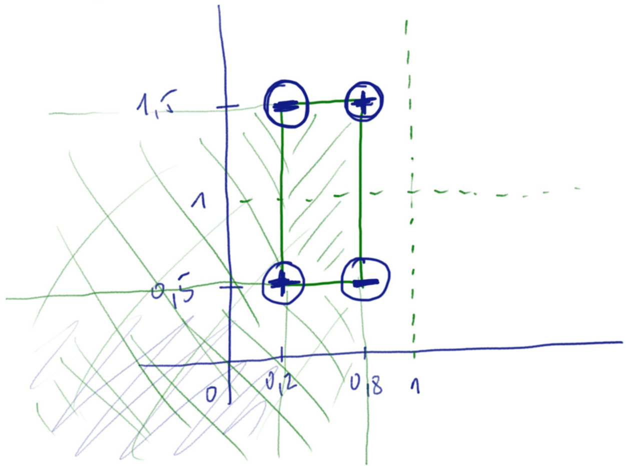
$(x,y) \in \langle 0,1) \times \langle 1,\infty)$:

$$F_{(X,Y)}(x,y) = \int_0^x \int_0^1 (u+v)\, dv\, du$$

$$= \int_0^x \left[uv + \frac{v^2}{2}\right]_0^1 du$$

$$= \int_0^x \left(u + \frac{1}{2}\right) du = \left[\frac{u^2}{2} + \frac{u}{2}\right]_0^x$$

$$= \frac{x^2+x}{2}$$



$P[0{,}2 \le X \le 0{,}8\ ;\ 0{,}5 \le Y \le 1{,}5]$

$$F_{(X,Y)}(0{,}8;1{,}5) - F_{(X,Y)}(0{,}2;1{,}5) - F_{(X,Y)}(0{,}8;0{,}5)$$

$$+ F_{(X,Y)}(0{,}2;0{,}5)$$