① $X \in \{0,1,2\}$

hodnota	0	1	2
prav.	a	b	a+b
pozorovaná četnost	13	16	21
prav.	a	$\frac{1}{2}-a$	$\frac{1}{2}$

$a, b \in \mathbb{R}$

$a+b+(a+b)=1$

$2a+2b=1$

$b=\frac{1}{2}-a$

$a+b=a+\frac{1}{2}-a=\frac{1}{2}$

1. Metoda momentů (MM)

$EX^2 = 0^2 a + 1^2\left(\frac{1}{2}-a\right) + 2^2 \cdot \frac{1}{2}$

$= \frac{1}{2} - a + 1 = \frac{3}{2} - a$

EX
EX^2
EX^3
;

$\overline{x} = \frac{0+0+\ldots+1+1+\ldots+2+2+\ldots}{50}$

$= \frac{0\cdot 13 + 1\cdot 16 + 2\cdot 21}{50} = \frac{58}{50}$

$\frac{3}{2} - \hat{a} = \frac{58}{50} \Rightarrow \boxed{\hat{a} = \frac{3}{2} - \frac{58}{50} = \frac{17}{50}}$

2. Metoda max. věrohodnosti (MMV)

likelihood $L(a) = \prod p(x_i; a)$

$L(a) = \underbrace{p(0)\cdot p(0)\cdot p(0)\ldots}_{13} \cdot \underbrace{p(1)\cdot p(1)\ldots}_{16} \underbrace{p(2)\,p(2)\ldots}_{21}$

$= p(0)^{13}\, p(1)^{16}\, p(2)^{21}$

$= a^{13}\cdot\left(\frac{1}{2}-a\right)^{16}\cdot\left(\frac{1}{2}\right)^{21} \in \langle 0,1]$

log-likelihood $\ell(a) = \ln L(a) = 13\ln a + 16\ln\left(\frac{1}{2}-a\right) + 21\ln\frac{1}{2}$

$\ell'(a) = \frac{13}{a} + \frac{-16}{\frac{1}{2}-a}$

$\ell'(a) = 0$

$\frac{13}{a} - \frac{16}{\frac{1}{2}-a} = 0$

$\frac{13}{2} - 13a - 16a = 0$

$\frac{13}{2} = 29a$

$\boxed{\hat{a} = \frac{13}{58}}$

2. X ... geometrické r. $p_X(x) = q^x(1-q)$ $q \in (0,1)$

x_i	0	1	2	3
n_i	29	16	4	1

1. MM

$EX = \frac{q}{1-q}$

$\overline{x} = \frac{29\cdot 0 + 16\cdot 1 + 4\cdot 2 + 1\cdot 3}{29+16+4+1} = \frac{24}{50}$

$\frac{\hat{q}}{1-\hat{q}} = \frac{24}{50}$

$50\hat{q} = 24 - 24\hat{q}$

$\boxed{\hat{q} = \frac{24}{74}}$

2. MMV

$L(q) = \underbrace{p(0)\cdot p(0)\cdot\ldots}_{29} \cdot \underbrace{p(1)\,p(1)\ldots}_{16}$

$= p(0)^{29}\, p(1)^{16}\, p(2)^{4}\, p(3)^{1}$

$= \left[q^0(1-q)\right]^{29}\cdot\left[q^1(1-q)\right]^{16}\cdot\left[q^2(1-q)\right]^{4}\cdot\left[q^3(1-q)\right]^{1}$

$= q^{16+8+3}\cdot(1-q)^{29+16+4+1}$

$L(q) = q^{24}\cdot(1-q)^{50}$

$\ell(q) = 24\ln q + 50\ln(1-q)$

$\ell'(q) = \frac{24}{q} + \frac{-50}{1-q} = 0$

$24 - 24\hat{q} - 50\hat{q} = 0$

$\boxed{\hat{q} = \frac{24}{74}}$

(3) spravná kostka S, falešná k. F

$X = \text{Mix}_{(1-w, w)}(S, F)$

	1 .. 5	6
$P[S=x]$	$\frac{1}{6}$	$\frac{1}{6}$
$P[F=x]$	$\frac{1}{10}$	$\frac{1}{2}$
$P[X=x]$	$(1-w)\frac{1}{6} + w\frac{1}{10}$	$(1-w)\frac{1}{6} + w\frac{1}{2}$

$EX = 3{,}5 = \frac{7}{2}$

1	2	3	4	5	6
11	10	14	10	18	34

1. mm:

$EX = (1+2+3+4+5)\left[(1-w)\frac{1}{6} + w\frac{1}{10}\right] + 6\cdot\left[(1-w)\frac{1}{6} + w\frac{1}{2}\right] =$

$EF = \underbrace{(1+2+3+4+5)}_{15}\frac{1}{10} + 6\cdot\frac{1}{2} = \frac{15}{10} + \frac{6}{2} = \frac{9}{2}$

$EX = (1-w)ES + wEF = (1-w)\frac{7}{2} + w\frac{9}{2}$

$\bar{x} = \frac{11\cdot 1 + 10\cdot 2 + 14\cdot 3 + \dots}{11+10+14+\dots} = \frac{425}{100} = \frac{17}{4}$

$(1-w)\frac{7}{2} + w\frac{9}{2} = \frac{17}{4} \quad /\cdot 4$

$14 - 14w + 18w = 17$

$4w = 3$

$\hat{w} = 3/4$

2. mmv:

$L(w) = p(1)^{11}\cdot p(2)^{10}\cdot p(3)^{14}\cdot p(4)^{10}\cdot p(5)^{18}\cdot p(6)^{34}$

$= \left[(1-w)\frac{1}{6} + w\frac{1}{10}\right]^{11+10+14+10+18}\cdot\left[(1-w)\frac{1}{6} + w\frac{1}{2}\right]^{34}$

$= \left[\frac{5-2w}{30}\right]^{63}\cdot\left[\frac{1+2w}{6}\right]^{34}$

$\ell(w) = 63\ln(5-2w) - 63\ln 30 + 37\ln(1+2w) - 37\ln 6$

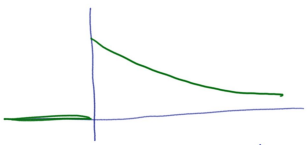
$\ell'(w) = \frac{-2\cdot 63}{5-2w} + \frac{2\cdot 37}{1+2w} = \frac{244 - 400w}{(5-2w)(1+2w)} = 0$

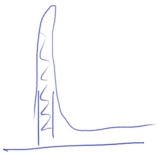
$244 - 400w = 0$

$\hat{w} = \frac{244}{400} = 0.61$

(6) X ... exponenciální r.

$f_X(x) = \frac{1}{\tau}e^{-\frac{x}{\tau}}; \quad \tau > 0$





0.87 1.28 0.9 0.34 0.25

1. mm: $EX = \tau$

$\bar{x} = \frac{3{,}67}{5} = 0{,}734$

$EX = \bar{x}$

$\hat{\tau} = 0{,}734$

2. mmv: $\Lambda(\tau) = \prod f(x_i; \tau)$

$\Lambda(\tau) = f(0{,}84)\cdot f(1{,}28)\cdot f(0.9)\dots$

$= \frac{1}{\tau}e^{-\frac{0{,}84}{\tau}}\cdot\frac{1}{\tau}e^{-\frac{1{,}28}{\tau}}\cdot\frac{1}{\tau}e^{-\frac{0{,}9}{\tau}}\cdot\frac{1}{\tau}e^{-\frac{0{,}34}{\tau}}\cdot\frac{1}{\tau}e^{-\frac{0{,}25}{\tau}}$

$= \frac{1}{\tau^5}e^{-\frac{0{,}84 + 1{,}28 + \dots}{\tau}} = \frac{1}{\tau^5}e^{-\frac{3{,}67}{\tau}} = \Lambda(\tau)$

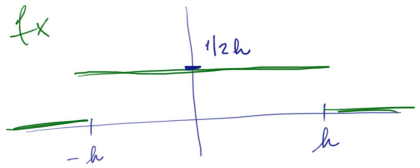
$\lambda(\tau) = -5\ln\tau - \frac{3{,}67}{\tau}$

$\lambda'(\tau) = -\frac{5}{\tau} + \frac{3{,}67}{\tau^2} = 0$

$-5\tau + 3{,}67 = 0$

$\tau = \frac{3{,}67}{5} = 0{,}734$

X ... rovnoměrné na $\langle -h, h \rangle$



(-4) -3 -2 -1,5 0,5 1 2,5 3

$$f_X(x) = \begin{cases} \frac{1}{2h} & \text{pro } x \in \langle -h, h \rangle \\ 0 & \text{jinak} \end{cases}$$

1. mm: $EX = \int_{-\infty}^{\infty} x f_X(x)\,dx = 0$

$$EX = \int_{-h}^{h} x \cdot \frac{1}{2h}\,dx = 0$$

$EX^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = DX$

$$EX^2 = \int_{-h}^{h} x^2 \cdot \frac{1}{2h}\,dx = \frac{1}{2h} \cdot \frac{1}{3}\left[x^3\right]_{-h}^{h} = \frac{2h^3}{2 \cdot 3 \cdot h} = \frac{h^2}{3}$$

$$m_2 = \frac{1}{n}\sum x_i^2 = \frac{16 + 9 + 4 + 2{,}25 + 0{,}25 + \dots}{8} = \frac{47{,}75}{8}$$

$$\frac{\hat{h}^2}{3} = \frac{47{,}75}{8} \qquad \hat{h} = \pm\sqrt{\frac{3 \cdot 47{,}75}{8}} \doteq 4{,}23$$

$h \geq 4$!

2. mm: $L(h) = f(-4) \cdot f(-3) \cdot f(-2) \cdots$

$$= \frac{1}{2h} \cdot \frac{1}{2h} \cdot \frac{1}{2h} \cdots \Rightarrow h = 4$$

$$= \left(\frac{1}{2h}\right)^8 \Rightarrow h = 0{,}2$$