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Technical Note

Water table response to a tidal agitation in a coastal aquifer: The Meyer–Polubarinova-Kochina theory revisited

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SUMMARY

The linear potential model of Meyer and Polubarinova-Kochina (MPK) solves Laplace's equation in a quadrant, the vertical face of which is subject to a harmonic variation of hydraulic head induced by a tide. On the horizontal face of the quadrant a linear combination of the partial derivatives of the head with respect to time and with respect to the vertical Cartesian coordinate are linearly related. The fields of pore pressure (head) and Darcian velocity, trajectories of marked particles, the oscillating phreatic surface and its upper envelope are obtained. Hydrograph in a piezometer located 60 m from the shore line in a thick unconfined coastal aquifer (Oman) is interpreted by the analytical results of the model using the detected groundwater amplitude attenuated as compared with the tide amplitude. MPK method provides a potentially superior description to the Dupuit–Forchheimer method for tidally driven phreatic systems.

1. Introduction

"*Time is infinite motion without a single instance of rest*". L. Tolstoy, War and Peace

Cyclostationarity, i.e. repetitiveness with certain periods, amplitudes and phase shifts is one of the cornerstones of modern science: orbiting planets in celestial mechanics, life in biology, Tolstoy's pendulum-sweeping civilization clashes and military invasions in geopolitics or the hydrological cycle in geosciences, among many other phenomena, are characterized by temporal regularity. Historically, processes with linear and non-linear growth-decay kinetics called for monomials and exponential functions, while the sin-cos-functions based description of harmonic (or anharmonic) behaviour of an entity exposed to "signals" ("agitations", "forcing") emerged as the Fourier series-integral analysis/synthesis. In groundwater hydrology, the most common natural, periodically excited system is a coastal aquifer, the head in which oscillates in response to the astronomic cyclicity (realized through easily observable tides). Simple-coupled harmonic variation of the ocean stage is imposed as a boundary condition in the contiguous porous layers, where the hydraulic head observations require special gadgets (piezometers, data-loggers).

Coastal aquifers in Oman (especially, in the Batinah region, Kacimov et al., 2009a,b) and in other arid countries (Pulido-Bosch et al., 2004), are important sources of both fresh groundwater from a terrestrial submarine discharge (if any) used for near-coast irrigation, and of encroached beach-bank-filtered sea water used as influent of desalination plants. Usually, a shallow (2–10 m) well is dug/drilled in an unconfined aquifer composed of sand or gravely unconsolidated sediments and either fresh water lens is skimmed or saline water is abstracted. These wells are located several meters/tens of meters from the shore line and therefore the tide effect on the water level in the wells is quite pronounced. Amplitudes of tides in the Gulf of Oman in the Batinah region are within the range of 1–1.3 m (see the link below).

The objective of this note is to retrieve a hydrograph from a shallow piezometer installed in an unconfined aquifer in the study area (Al-Hail site, Samail catchment, Southern Batinah), to link this hydrograph to the tide amplitude and to interpret the results using the Linear Potential Theory (LPT) of Meyer (1955-1956) and Polubarinova-Kochina (1959; Polubarinova-Kochina and Kochina, 1994), abbreviated as MPK below and recently implemented on exactly the same site for modeling artificial recharge experiments (Kacimov et al., 2009b).

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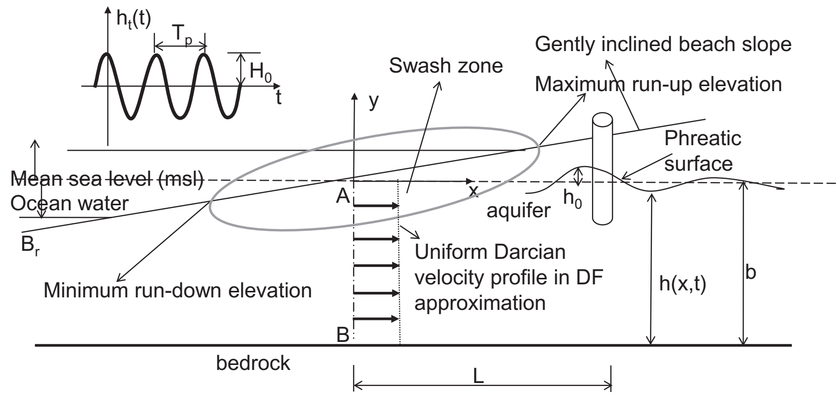

Fig. 1. Vertical cross-section of a finite-thickness unconfined coastal aquifer.

For coastal aquifers (if not fjordic and not interspersed by river channels perpendicular to the shore line) on a moderate scale of this study a 2-D flow obeys the Darcy law:

$$\vec{V} = -k\nabla h(x, y, t) \quad (1)$$

where x and y are the landward-oriented and vertical Cartesian coordinates (Fig. 1), t is time, k is hydraulic conductivity (constant) and the Darcian velocity vector $\vec{V}(x, y, t)$ has components u and v in x and y directions, respectively.

In this note we are not interested in the swash, riparian and near-shore zone (an oval demarcates this zone in Fig. 1) where infiltration-exfiltration – although also cyclostationary and monitorable by standard piezometers – is controlled by the beach properties (e.g., Austin and Masselink, 2006) and is important for beach morphology, sediment transport in the surf zone, beach biota, near-shore sea currents and wave geomechanical impact/energy transfer through the beach skeleton (see, e.g., Finn et al., 1983; McLachlan and Brown, 2006). Our focus will be on the part of the phreatic surface located relatively far from the shore line, i.e. where high-frequency fluctuations of the sea level are already filtered by a porous beach cushion. At the same time, the piezometer, where water table fluctuations are observed, should not be too far from the beach because the amplitude of these fluctuations decreases with x. Moreover, at high x in Fig. 1 other hydrological effects (e.g. irrigation, traffic load, density contrast close to the interface of a sea water intrusion zone, Kacimov et al., 2009a, etc.) interfere and obliterate the tidal agitation.

Since the Ferris (1951)–Jacob (1950) papers and earlier Dutch contributions (see De Ridder and Wit, 1965 for references) the effect of tides on unconfined and confined aquifers and mathematically equivalent problems of surface waves impacting earth dams have been studied theoretically, in laboratory experiments and in the field (e.g., Carr and Van Der Kamp, 1969; De Cazenove, 1971; Edelman, 1972; Erskine, 1991; Fakir and Razack, 2003; Geng et al., 2009; Kim et al., 2007; King et al., 2010; Li et al., 1999; Mishra and Jain, 1999; Ojima, 1977; Ogris, 1972; Reynolds, 1987; Roberts et al., 2010; Rotzoll et al., 2008; Teo et al., 2003; Trefry, 1999; Vandenbohede and Lebbe, 2007; Wang and Tsay 2001; Zhou, 2008), with a recent focus on variable-density and solute transport numerical codes (e.g., Li et al., 2009). The Dupuit–Forchheimer (DF) approximation, usually exploited in models describing the groundwater zone inland (and far from the oval in Fig. 1) assumes a horizontal (quasi-horizontal) piston-type cyclostationary motion of water from/into an open water body (e.g., sea, reservoir, river, etc. or their combinations demarcating a porous massif in the plane not shown in Figs. 1 and 2) into/from an adjacent aquifer of a certain thickness, k and storage coefficient. The amplitude and phase of water level fluctuations in piezometers (observational wells) are related to the inducing signal from the source of cyclic agitations that is often used in solving inverse problems.

Since Putnam (1949) and Reid and Kajiura (1957) it is well understood that a cyclic excitation of Darcian flows of a constant-density incompressible fluid in an incompressible fully-saturated porous skeleton subjacent to any surface water waves should be described by the Laplace equation in the (x, y)-plane. If a flat porous sea floor is subject to periodic wave impacts from above (as in Putnam–Reid–Kajiura schemes and numerous papers published after them, see, e.g. Packwood and Peregrine, 1980), then solving a boundary-value problem for this equation is mathematically similar to the steady model of groundwater motion elucidated by Tóth (1963) (see, e.g., Bokuniewicz, 1992), i.e. the theory of holomorphic functions can be used for these two different applications (in coastal engineering, e.g. Magda, 2000 and in hydrogeology, e.g. Craig, 2008).

If a porous massif has a phreatic surface(s), as in coastal aquifers or porous embankments, then, generally speaking, a non-linear condition should be satisfied on this free boundary (Roberts et al., 2010; Polubarinova-Kochina and Kochina, 1994). The mathematical complexity of the problem in Fig. 1 is exacerbated by the presence of a transient seepage face, periodically appearing and disappearing on a beach slope or even on both sides of a depression periodically emerging on the lee side of a beach bank crescent (Aitchison et al., 1983). During the exfiltration stage, decoupling occurs between the groundwater phreatic surface and free surface in the reservoir free water body (Nielsen, 1990; Turner et al., 1996), which is actually the main stoss-side decoupling. Lee-side decouplings and seepage faces are short-lived and depend on the presence of microtopogrpahy depressions but hey are themselves "decoupled" with the stoss-side decoupling. The Laplace equation based, non-linear and transient phreatic-surface dynamics juxtaposed with, say, ocean waves (Nielsen et al., 1997) or river bores – crucial in the swash zone (oval in Fig. 1) – to the best of our knowledge, has not yet been described in a closed form, although several analytical approximations (see, e.g., Longuet-Higgins, 1983, who actually ignored the water table and solved the Laplace equation in a half-plane with a Dirichlet boundary condition for u, given on the beach surface), in particular, the perturbation analysis of Teo et al., 2003 and Roberts et al., 2010 (they took into account finite-slope beaches and obtained series expansions for several harmonics), and numerical codes (e.g., Gardner, 2005) have been implemented.

As numerical methods can tackle PDEs even more parametrically complex than the Laplace equation, the effects of capillarity/partial saturation were included through upgrading the governing equation to the Richards one, which has been tackled

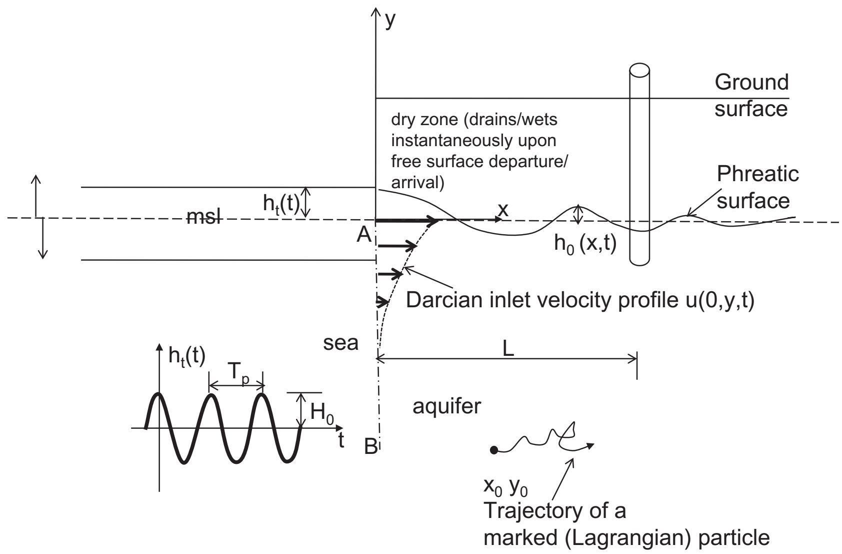

Fig. 2. Vertical cross-section of an infinite-thickness unconfined coastal aquifer.

by FDM–FEM–BEM. Incorporation of the capillary fringe and vadose zone in the tidal effect model exterminates the free boundary but demands the soil water retention and unsaturated permeability functions and additional boundary conditions on the soil–air interface, that is seldom feasible in the field.

It is noteworthy that the topology of the resulting seepage streamlines, reproduced from solving either the Laplace or Richards equation, resembles ones induced by vortices (perpendicular to a vertical plane of Fig. 1), with "hinge lines" similar to ones of Tóth (1963), although, of course, vorticity in commonly posited constant-density fluids is nil. These "hinge lines" are transient, with an exception of standing waves on a horizontal sea floor, i.e. the loci of the "hinge lines" move up and down along the beach contour (or sea floor) with the wave set up as in e.g., bores, tides, higher-frequency waves (Ataie-Ashtiani et al., 2001; Li and Barry, 2000), or traveling oceanic waves.

2. Boundary-value problems and their explicit solutions

2.1. Dupuit–Forchheimer approximation

The Boussinesq equation (Baird et al., 1998; Horn, 2002) in an unconfined horizontal homogeneous isotropic aquifer of an effective porosity s (also called drainable porosity, specific yield-specific storage in no-hysterisis media, storage coefficient; not to confuse with specific storage in confined aquifers), with a horizontal subjacent bedrock and semi-infinite in the positive x direction (Fig. 1) is written for the saturated thickness $h_s(x, t)$ of a fluid sandwiched between the bedrock and phreatic surface:

$$k\frac{\partial}{\partial x}\left(h_s\frac{\partial h_s}{\partial x}\right) = s\frac{\partial h_s}{\partial t} \tag{2}$$

i.e. in Eq. (2) the y-coordinate is effectively exterminated from Eq. (1). Several generalisations of PDE (2) to sloping bedrocks, leaky aquifers stacked beneath one where h_s is searched and recharge on the phreatic surface have been examined for cyclostationary boundary conditions (e.g., Smith, 2008; Su et al., 2003).

No explicit analytical solution for a boundary condition corresponding to a tidal excitation (even for a simple harmonic) are known for Eq. (2). Integral estimates are, however, possible. In this manner, Philip (1973) predicted the effect of overheight (superelevation) of the period-averaged water table above the mean sea level (msl), later confirmed in laboratory and field experiments, as well as in various perturbation solutions by Knight (1981), Li et al. (2000), Parlange et al. (1984), Roberts et al. (2010), Song et al. (2006, 2007), Teo et al., 2003, Zyryanov and Khublaryan (2006). As Kacimov (2002) argued in congruity with Philip's (1973) seminal ideas, superelevation is a generic property of a non-linear diffusion equation and is not caused by a seepage face (contrary to what some of Philip's disciples/adversaries stated).

The lack of analytical solutions to (2) doomed practical groundwater hydrologists to various types of linearisations (Polubarinova-Kochina and Kochina, 1994), e.g. to representation of h_s in (2) as $h_s = b + h_0(x, t)$, where b is the "average" saturated thickness. Then putting this sum into (2) and by ignoring the term $(\partial h_0/\partial x)^2$ – that is allegedly true because the slope of the water table is already postulated to be small in the original non-linear Boussinesq equation (2) – the governing PDE is simplified to:

$$\frac{\partial^2 h_0}{\partial x^2} = \frac{s}{T}\frac{\partial h_0}{\partial t} \tag{3}$$

where $T = kb$ is "transmissivity". Eq. (3) makes most groundwater hydrologists happy because for this linear PDE the whole analytical machinery, from, e.g., the Carslaw and Jaeger, 1959 heat conduction compendium, becomes available. If h_0 in Eq. (3) is counted from msl, then the tide-related problem is solved with the boundary conditions

$$h_0(\infty, t) = 0, \quad h_0(0, t) = h_t(t) \tag{4}$$

In Eq. (4) the second boundary condition, determined by the tide, is imposed on a vertical line AB (dashed-dotted line in Fig. 1). The real sea floor, AB_r, is usually gently sloped and, hence, far from AB. This deviation of the contours, through which agitation is induced, is, in our opinion, one of the major flaws of the DF theory (2) and its linearisation (3).

From now on we assume the simplest signal characterized by a single harmonic:

$$h_t = H_0\cos\frac{2\pi t}{t_p} \tag{5}$$

where H_0 is the amplitude of the tide and t_p is the period. Then the so-called Ferris (1951)–Jacob (1950) solution[1] to the boundary-value problem (3)–(5) is

$$h_0 = H_0 \exp\left[-\sqrt{\frac{\pi s}{t_p T}}x\right]\cos\left(\frac{2\pi t}{t_p} - x\sqrt{\frac{\pi s}{t_p T}}\right) \quad (6)$$

We emphasize that in Eq. (6) three independent physical characteristics of an unconfined aquifer are imbedded: *s, k* and "average" saturated thickness *b* ($T = kb$). How to get the latter is a special story (Polubarinova-Kochina and Kochina, 1994).

2.2. Linear potential theory

Half a century ago, MPK developed an alternative (alas, well-forgotten) theory for predicting water table oscillations in an unconfined aquifer. In 1955–1956 Meyer published a series of papers where a purely cyclostationary tidal excitation was addressed (with no initial conditions). Polubarinova-Kochina (1959) considered a more general case of an arbitrary initial condition for a phreatic surface (e.g. a given steady state regime preceding the further transient "agitation") disturbed by an arbitrary reservoir stage variation (in particular, by a harmonic agitation). From the Polubarinova-Kochina (1959) solution the results of Meyer follow as a particular case (large-time limit when the "memory" of the initial water table shape is effaced). We shall base our analysis on the latest derivations of Polubarinova-Kochina and Kochina (1994) who considered a porous quadrant $x > 0$, $y < 0$ (Fig. 2), fully saturated (capillarity ignored). The Darcian velocity potential $\varphi(x, y, t) = -kh(x, y, t)$ obeys the Laplace equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad (7)$$

A non-linear condition holds:

$$s\frac{\partial\phi}{\partial t} + k\frac{\partial\phi}{\partial y} + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = 0 \quad (8a)$$

on an unknown free boundary. Eq. (8a) has recently caused a hot discussion (Strack and Verruijt, 2010) which showed that even experienced groundwater hydrologists stumble on its interpretation. This equation is as nasty as all non-linear wave problems (Stoker, 1957) and even a step-function type solution (a sudden rise of the water level in a reservoir, De Wiest, 1960) or similarity solution (constant-rate rise of the water level in two adjacent reservoirs, Kacimov and Yakimov, 2001) are mathematically so tedious that (8a) is commonly linearised to

$$\frac{\partial\phi}{\partial t} + \frac{k}{s}\frac{\partial\phi}{\partial y} = 0 \quad (8b)$$

which is imposed on the fixed $y = 0$ line (see Ogris, 1972 for generalization of Eq. (8b) to seepage counting for inertial effects, usually ignored in standard groundwater flow models). The initial condition for (7) is taken as:

$$\phi(x, y, 0) = 0 \quad (9)$$

which means that at $t = 0$ all groundwater is at rest. Other conditions on the boundaries of the porous quadrant in Fig. 2 and far from the beach-free surface are:

$$\phi(0, y, t) = -kh_t, \quad c\phi(x, 0, t) = -ky, \quad \phi \to 0 \quad \text{at} \quad r \to \infty, \quad r = \sqrt{x^2 + y^2} \quad (10)$$

The last asymptotics in Eq. (10) implies no regional or pumping-induced flow far from the shoreline.

Polubarinova-Kochina and Kochina's (1994) solution in terms of *h* is:

$$h(x, y, t) = -\frac{2}{\pi}h_t(t)\arctan\frac{y}{x} + \frac{2kx}{s\pi}\int_0^t \frac{h_t(t-u)}{(ku/s - y)^2 + x^2}du \quad (11)$$

The phreatic surface is determined from Eq. (11) using the isobaricity condition following from the second equation in (10) as:

$$h_o(x, t) = h(x, 0, t) \quad (12)$$

MPK model has its own deficiencies: it ignores the unsaturated zone and capillary fringe, seepage face and decoupling in the oval zone of Fig. 1, a clogging thin layer occurring on the interface between the ocean water and interface (pretty easily countable in the DF theory) makes the LPT problem mathematically intractable, an arbitrary beach slope in Fig. 1 is difficult to tackle, linearization – as tested against explicit solutions in terms of the non-linear potential model for decay of groundwater mounds (Polubarinova-Kochina, 1977) – introduces errors if wave amplitudes are high, compressibility is neglected. We recall that LPT is valid if L of the piezometer in Fig. 1 is greater than the abscissa of the point of maximum run-up elevation (whether this run-up is caused by a tide, swell, bore or any other variation of the sea level). In other words, the phreatic surface fluctuations $h_0(x, t)$ in Fig. 2 are not disturbed by the groundwater hump dynamics near the run-up where the effects of the seepage face and asymmetry of infiltration-exfiltration are indeed important.

As one can see from (7), (8a), (8b), (9), (10), (11), (12), LPT is an essentially 2-D (in a vertical plane) "deep-water-theory" (see also Brock, 1976; Carravetta, 1957; Dagan, 1964; Hunt, 1970; Polubarinova-Kochina and Kochina, 1994; Rowan, 1957; Stoker, 1957; Tyvand, 1984), which does not pre-impose any verticality of constant head lines (as the DF approximation does). DF theory assumes that along *AB* in Fig. 1 the magnitude of the Darcian velocity, $|\vec{V}|$, does not vary with *y*. The vertical component v of $\vec{V}$ is either ignored in the "standard" DF theory, or varies linearly with *y* in the "generalized hydraulic" approximation of Polubarinova-Kochina and Kochina (1994). It is, however, proved (see e.g., Ataie-Ashtiani et al., 2001; Cartwright et al., 2006; Gardner, 2005) that v can be significant in tidally excited unconfined aquifers; close to the free surface seepage is "much" (and nonlinearly!) stronger than at the base of even relatively thin aquifers.

In LPT the distribution *u(y)* along the vertical beach in Fig. 2 is a part of solution and this function can be even non-monotonic (Kacimov, 2009b) that is totally at odd with the "hydraulic" principle. The profile $|\vec{V}|$, shown in Fig. 2, immediately follows from Eq. (11) by differentiation. The direction of $\vec{V}$ along *AB* is, of course, alternating (with time) from landward to seaward according to the imposed harmonic (5).

If we put (5) as the integrand into (11), then an explicit expression for *h(x, y, t)* is delivered by Wolfram's (1991) *Mathematica* in terms of combinations of the CosIntergal, SinhIntegral and SinIntegral functions, i.e. *Mathematica* routines. A routine differentiation of *h* with respect to *x* and *y* gives explicit expressions for *u(x, y, t)* and *v(x, y, t)*. We drop all these lengthy expressions for the sake of brevity. Fig. 3 shows the contour plots $h(x, y, 1.1) = -0.1, -0.01, 0.05, 0.1, 0.2, 0.3$ (curves 1–6, correspondingly) for $H_0 = 1.1$ m, $t_p = 0.5$ day, $s = 0.2$, $k = 30$ m/day (these four values were fixed in all further calculations; in the next Section we will explain why). Fig. 4a shows a snapshot of the vertical distribution

[1] Clearly, like the Theis confined-aquifer - sink-well solution, (6) was well known to engineers dealing with heat conduction, see Carslaw and Jaeger, 1959, Chapter II, Section 6 and viscous flows of Newtonian fluids, decades prior to the Ferris and Jacob publications. The original ideas can be tracked back to the Poisson treatise: Memoire sur la distribution de la chaleur dans les corps solides. J. de l'e Polytechnique. Paris, 1823.

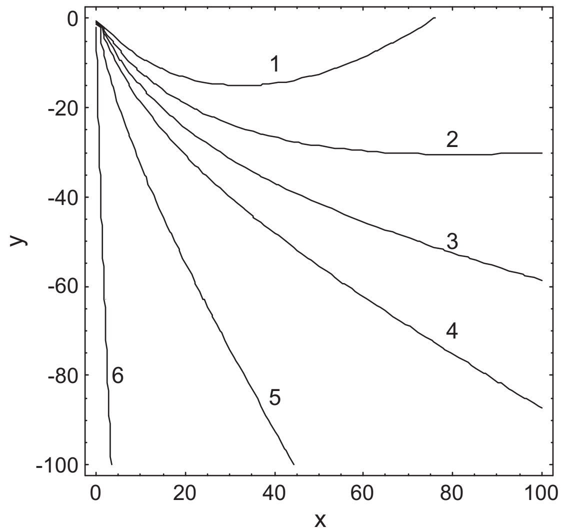

Fig. 3. Contour plots $h(x, y, 1.1) = -0.1, -0.01, 0.05, 0.1, 0.2, 0.3$ (curves 1–6, correspondingly) for $H_0 = 1.1$ m, $t_p = 0.5$ day, $s = 0.2$, $k = 30$ m/day.

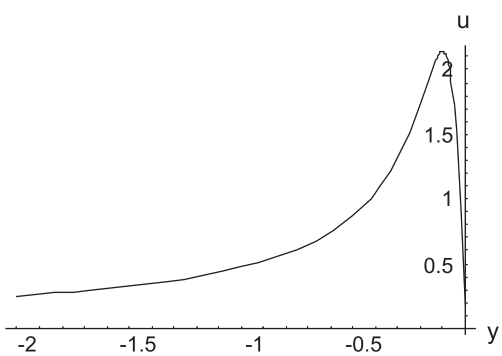

Fig. 4a. Snapshot of the $u(y)/30$ at cross-section $x = 0.1$ and time instance $t = 1.1$.

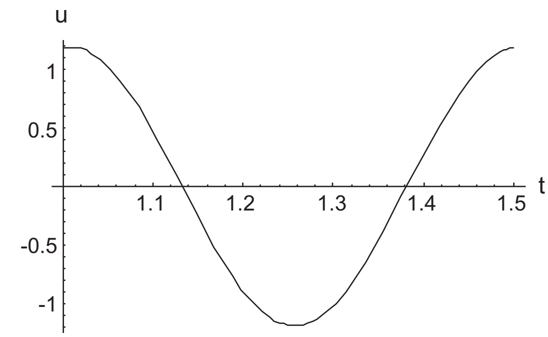

Fig. 4b. Dimensionless horizontal velocity $u(t)/30$ at an observational point $x = 0.1$, $y = -1$.

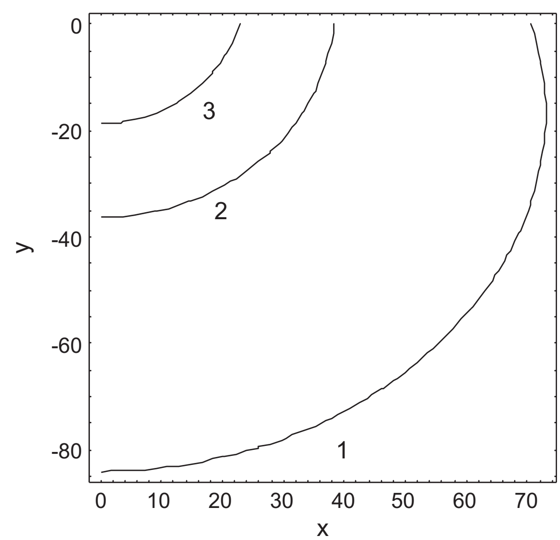

Fig. 5. Contour plot of the Darcian velocity modulus $|\vec{V}|(x, y)$ at $t = 1.1$ (curves 1–3 correspond to $|V| = 0.1, 0.3, 0.7$).

of horizontal velocity related to conductivity, $u(0.1, y, 1.1)/30$. In Fig. 4b the time-graph of $u(0.1, -1, t)/30$ is plotted. Fig. 5 shows the contour plot of the magnitude of $|\vec{V}|(x, y, 1.1)$ (curves 1–3 correspond to $|\vec{V}| = 0.1, 0.3, 0.7$). As one can see from Figs. 4a and 5, the DF approximation is totally inadequate.

We note that close to the shore line the LPT-predicted magnitude of the hydraulic gradient exceeds the Polubarinova-Kochina critical limit of 1. Consequently, dislodging of fine fraction particles in the beach matrix can occur, with translocation of these particles to the sea during the half-cycle when $u < 0$ at $x = 0$. This seepage (exfiltration) induced erosion determines the steepness of beaches (McLachlan and Brown, 2006). Additionally, the locus of high gradients in the quadrant near the origin of coordinates of Fig. 2 may cause deviations from the postulated Darcy law.

As in Kacimov et al. (1999), we reconstruct the trajectories of marked particles using the system of two non-autonomous nonlinear ODEs:

$$\frac{dx_p}{dt} = \frac{1}{m} u\big(x_p[t], y_p[t], t\big), \quad \frac{dy_p}{dt} = \frac{1}{m} v(x_p[t], y_p[t], t) \tag{13}$$

where m is porosity (for simplicity we assumed that $m = s$) and (x_p, y_p) are the Lagrangian coordinates of a selected particle.[2] The system (13) is solved with initial conditions $(x_p, y_p) = (x_0, y_0)$ at $t = 0$ by the NDSolve (Runge–Kutta) *Matheamtica* routine. Fig. 6 shows the loops of the particle trajectory starting its journey at $t = 0$ from for $x_0 = 7$, $y_0 = -5$ with marching time of 3 days, plotted by the ParametricPlot routine of *Mathematica*. Obviously, the reconstructed kinematics of marked particles should be used with caution: if a particle trajectory crosses the lines $x = 0$ or $y = 0$, then time-marching should be stopped, although mathematically the dynamic system (13) can be integrated for any $t > 0$.

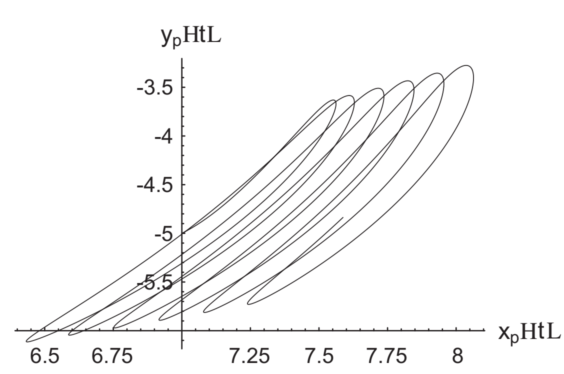

Fig. 6. Trajectory of a marked particle initially located at $x_0 = 7$, $y_0 = -5$.

[2] We recall that u and v in eqn. (13) are explicitly expressed through *Mathematica* special functions i.e. no numerical differentiation of the nodal head values, as in most numerical codes, is involved.

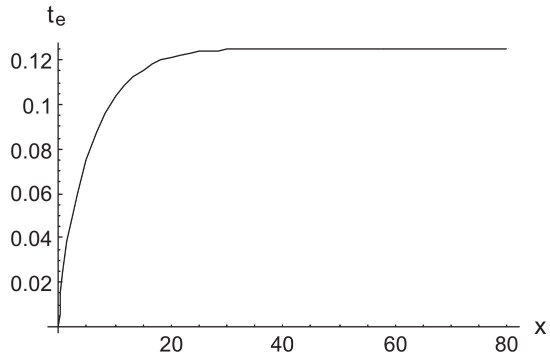

Fig. 7a. Time of the water table peak closest to the peak $t = 0$ of the sea level.

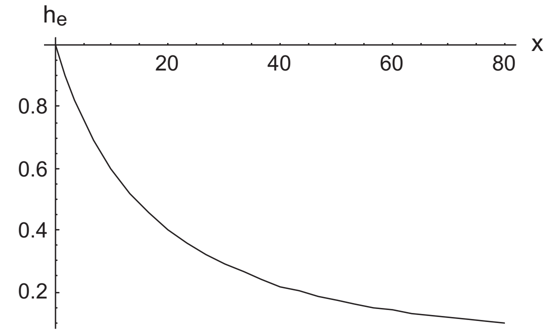

Fig. 7b. Water table upper envelope.

The velocity field and trajectories are needed in environmental applications when the fate of contaminants agitated by the tide and migration of chemicals from the aquifer into the sea (and back) are important (e.g., Ataie-Ashtiani et al., 2001; Staver and Brinsfiled, 1996).

The water table equation is obtained from general Eqs. (11) and (12). At relatively large t, when the memory of the initial ($t = 0$) flat water table vanishes, the MPK water table equation reads:

$$h_0(x,t) = -H_0\left[\exp\left(-\frac{2\pi s}{kT_p}x\right)\cos\frac{2\pi t}{T_p} + \frac{1}{\pi}N(x)\sin\frac{2\pi t}{T_p}\right],$$
$$N(x) = \frac{2xs}{k}\int_0^\infty \frac{\sin(2\pi u/T_p)}{u^2+(xs/k)^2}du \tag{14}$$

Using Wolfram's *Mathematica* routine Integrate we can write the integral in (14) in the form:

$$N(x) = \sqrt{\pi}\text{MeijerG}\left[\left\{\left\{\frac{1}{2}\right\},\{\}\right\},\left\{\left\{\frac{1}{2},\frac{1}{2}\right\},\{0\}\right\},\left(\frac{\pi s x}{k t_p}\right)^2\right] \tag{15}$$

where MeijerG is another *Mathematica* routine (Meijer's G-function; the Dutch mathematician C. Meijer introduced the generalization of the hypergeometric function in his paper: Meijer (1936)). In our computations, numerical integration by the NIntegrate routine of *Mathematica* in Eq. (14) gave practically the same results as MeijerG.

The time t_e between the tide peak in the ocean and a maximum of the hydrograph in the corresponding piezometer is obtained as solution of the equation:

$$\frac{\partial h_0(x,t)}{\partial t} = 0 \tag{16}$$

at a given locus $x = L$. Differentiating Eqs. (14) and (15) this time is found as:

$$t_e = \frac{t_p}{2\pi}\text{ArcTan}\left[\frac{\exp\left(\frac{2\pi s L}{kT_p}\right)}{\sqrt{\pi}}\text{MeijerG}\left[\left\{\left\{\frac{1}{2}\right\},\{\}\right\},\left\{\left\{\frac{1}{2},\frac{1}{2}\right\},\{0\}\right\},\left(\frac{\pi s x}{k t_p}\right)^2\right]\right] \tag{17}$$

We put t_e back into Eq. (14) and get the curve of all maxima of the phreatic surface, $h_0(x, t_e(x))$, the so-called water table upper envelope (water logging curve). Above this curve the porous medium is never saturated. In the Batinah region of Oman this curve will control the roots of irrigated coastal crops (plants), which do not tolerate the salinity of "upwelled" h_0 in Fig. 1. The curves $t_e(x)$ and $h_e(x) = h_0(x)/H_0$ are shown in Fig. 7a and 7b, respectively. We can see from Fig. 7a that $t_e(x)$ is not a linear function of x – contrary to what the DF theory (Eq. (6)) predicts.

3. Experiments and their LPT interpretation

We used a piezometer located about 60 m from the shore line in Al-Hail area which lies in the lower reaches of Samail catchment to the north of the capital Muscat, Oman (Kacimov et al., 2009a). The study area is characterized by thick accumulation of alluvium (100–600 m) (Macumber, 1997) that comprises the main source of groundwater in the area. The alluvium is predominantly composed of unconsolidated gravels with minor sand and silt formed mainly during the Quaternary. It is poorly sorted with large degree of heterogeneity. The piezometer penetrates a shallow depth (about 2.5 m counted from the ground surface) and reports an average water level at a depth of 1.8 m from the ground surface. In our tide-detection experiment we measured fluctuations in water table by using CTD-Diver model DI265 which is a data-logger that determines the height of a water column by measuring the water pressure with a built-in pressure sensor. Air pressure was measured simultaneously using a Baro-Diver, Model DI500. The diver was tighten to a rope and inserted into the piezometer below the water surface at an elevation of 400 cm that was taken as a reference point for measurements. The Diver measured the height of the water column above the Diver's pressure sensor (400 cm). The Diver was programmed to take measurements every 5 min for a full day cycle of measurements. The experiment began on 20/02/2008 at 09:16:29 and terminated on 21/02/2008 at 10:11:29.

Fig. 8 a shows absolute pressure in millibars retrieved by the Diver. Fig. 8b is a barometrically corrected (see Rasmussen and Crawford 1997 for the details on this effect) water level (in cm) in the peizometer i.e. water elevation above the data-logger itself. From Fig. 8b we can see that the amplitude of water table fluctuations in our piezometer is about 12 cm. Prior to the February-2008 continuous readings by Diver, in 2004–2007 we regularly measured the water table depth in these piezometers and detected the amplitude of fluctuations of the same magnitude i.e. $H_0 = 10$–12 cm is common.

We retrieved the ocean levels from the nearby station (23.6167°N, 58.6000°E) in Muscat (http://www.mobilegeographics.com:81/locations/3974.html?y=2008&m=2&d=20).with high-low tide values: February 20, 2008 8:55 AM 2.51 m, 3:16 PM 0.56 m, 10:06 PM GST 2.88 m., February 21, 2008 21 3:58 AM 1.23 m, 9:42 AM 2.57 m. We assumed that the tide generates a monochromatic semi-diurnal signal with $H_0 = 1.1$ m, and $t = 0.5$ days (12 h) i.e. we ignored a slight difference in amplitudes and periods of semi-diurnal components of the real tide.

We assumed that the bedrock is at infinity and that the specific yield is $s = 0.2$ that is confirmed in the study area by numerous pumping tests conducted by the Ministry of Regional Municipalities and Water Resources and with what we actually soil-sampled in 2003 in the 3-m deep pedon where later our piezometers (tap-

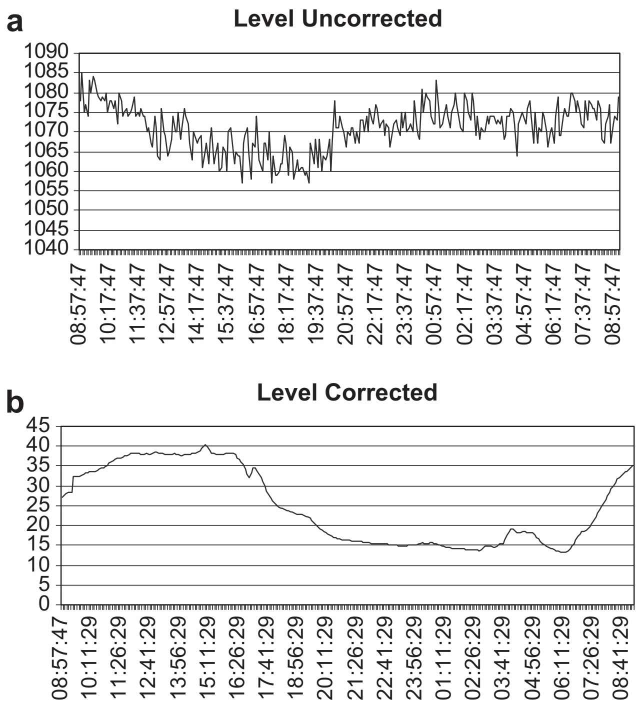

Fig. 8. Piezometer water levels, uncorrected (a) and corrected to the barometric effect (b).

ping just the top of the unconfined aquifer) were installed. As we have mentioned, there is no LPT-solution for a real gently inclined bed and we assumed a vertical (*AB*) agitation ray in Fig. 2. Then using (14), (15) we matched the amplitude found from LPT and from experiments (Fig. 8a and b) by varying *k*. The best fit is attained at $k \approx 30$ m/day. The found value of *k* is well within the range 8–70 m/day reported by the Ministry for this aquifer.[3] If we use the DF solution (6) for the whole possible range of reported aquifer thicknesses (100–500 m) in the whole Batinah region, then for the same *s* we get the amplitude of at least $H_0 \propto 0.3$–0.6 m that is significantly higher than what was observed in our piezometer.

We note that if A*x* in Fig. 2 is assumed to be a rigid caprock and the aquifer (confined by this ray) is infinite in –*y* direction, then in an elastic medium the governing equation for the hydraulic head *h*(*x*, *y*, *t*) is:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S_s}{k}\frac{\partial h}{\partial t} \tag{18}$$

where S_s is now specific storage, $S_s = \rho g(\alpha + m\beta)$, ρ is water density, *g* is gravity acceleration, α and β are skeleton and water compressibilities. Let us impose on *AB* of Fig. 2 the same boundary conditions as in the LPT theory i.e.

$$h(0, y, t) = H_0 \cos\frac{2\pi t}{t_p}, \quad h(\otimes\infty, \oplus\infty, t) = 0 \tag{19}$$

where $\otimes\infty, \oplus\infty$ are the signs of "or-infinities" in Cartesian coordinates. With the condition of rigidity of the caprock ($y = 0$), i.e. $h_y(0, x, t) = 0$ solution of (18) and (19), obviously, does not depend on *y* and is given by a transformed Eq. (6):

$$h(x, t) = H_0 \exp\left[-\sqrt{\frac{\pi S_s}{t_p k}}x\right] \cos\left(\frac{2\pi t}{t_p} - x\sqrt{\frac{\pi S_s}{t_p k}}\right) \tag{20}$$

[3] We emphasize that this k is "apparent" or "effective" value, integrating the hydraulic conductivity of the whole "participating" porous volume between AB in Fig. 2 and our piezometer.

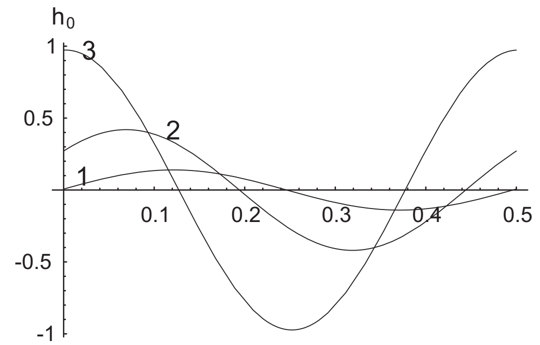

Fig. 9. Water table cycling at $x = 60$ m, $t_p = 0.5$ day and $k = 30$ m/day shown as dimensionless values h_0/H_0 (t) according to MPK Eq. (14) with $s = 0.2$, Eq. (20) with $S_s = 0.001$ m^2/N and Eq. (20) with $S_s = 10^{-6}$ m^2/N, correspondingly.

At no realistic values of S_s and *k* our piezometer hydrograph can be matched by (20). Fig. 9 illustrates this for $x = 60$ m, $t_p = 0.5$ day and $k = 30$ m/day. Curves 1–3 there show h_0/H_0 according to MPK Eq. (14) with $s = 0.2$, Eq. (20) with $S_s = 0.001$ m^2/N and Eq. (20) with $S_s = 10^{-6}$ m^2/N, correspondingly (the last two curves represent the upper and lower bounds of practical clay and sand-gravel skeleton compressibilities).

The above said recurs to the Bouwer (1978) admonishment against unscrupulous applications of the DF theory (see also Kacimov et al., 2009b) – now for tidally excited coastal unconfined aquifers. Namely, if the thickness of an unconfined aquifer is relatively high and disturbances of the head are prevalently from the upper part of the aquifer (e.g., Bouwer's groundwater mounds or our tides, hydraulically different from horizontal-piston-type agitations), then the flow topology (e.g., streamlines, streaklines, isobars, isotachs, isochrones, Poincare sections, etc.) may be completely different from what the DF theory postulates.

4. Conclusions

Linear potential theory of Meyer and Polubarinova-Kochina is employed for modeling the dynamics of groundwater in an unconfined aquifer of the Batinah region in Oman. The hydrograph in a shallow piezometer is used in conjunction with the tide data for assessing the hydraulic conductivity of a deep aquifer of a given effective porosity with no spurious "transmissivity" involved. The theory predicts 2-D transient variation of the head expressed as an integral, whose kernel is of the Poisson type and the integrand is the boundary condition on a vertical interface between the sea and a porous massif. As compared with the integral representation of solution to the boundary-value problem (3), (4), where the integrand has the kernel $(t-u)^{-3/2}\exp\left[-\frac{sx^2}{4(t-u)k}\right]$ and the boundary function is involved through $h_t(u)$, in LPT Eq. (11) involves $[(ku/s - y)^2 + x^2]^{-1}$ as the kernel but the agitation condition as $h_t(t-u)$. This becomes important if (11) at $y = 0$ is considered as an integral equation of the first kind, the left hand side of which is obtained from Diver-type piezometric observations and h_t (or even the planet motion in the Solar system) is to be reconstructed from this observation. LPT and field observation at an Eulerian point (piezometer) of the motion of Lagrangian groundwater particles enable a better understanding of a coastal (ocean-aquifer) systems.

In this note, the head (pore pressure), Darcian velocity and its components for a monochromatic (cosinusoidal) signal are obtained in an explicit and rigorous form. The dynamics of marked particles subject to these cyclostationary head fluctuations in a porous quadrant is obtained from a system of ODEs.

Why have the Ferris–Jacob formulae prevailed over MPK's ones, even in aquifers and for transient regimes where the very premises of the DF ("hydraulic") approximation are arrantly inappropriate? We deem that this was caused by the mathematical instrumentation available to the hydrologists at the time: Eqs. (6) and (20) involving elementary functions can be muddled through by any "practitioner" while the special functions (Sin–Sinh–Cos–Integrals, Meier's G-function) involved in the LPT-solution (see, e.g., Eqs. (14) and (15)) to the same problem need a more intricate platform (mathematical background and computer). With the advent of Wolfram's *Mathematica* and other computer algebra packages, the situation has however changed: practical hydrologists can now use "new" special functions, integrals and ODEs involving them in the well "hydraulics" as easily as the "old" Theis–Hantush special functions.

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