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Theoretical evaluation of non-uniform skin effect on aquifer response under constant rate pumping

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Abstract

Under field conditions, well skin may be non-uniformly distributed over the screen section. To investigate such non-uniform skin effect on aquifer response, a model is developed where the non-uniform skin effect is represented by an arbitrary piecewise continuous skin function $S_k(z)$ imposed on the boundary of the pumping well. Wellbore storage is taken into account in the pumping well. Due to $S_k(z)$ the model solutions are in terms of non-orthonormal functions, and the Gram–Schmidt method is employed to determine them. It is found that the wellbore flux distribution of the pumping well $Q_w(z, t)$ is inversely related to the variation of $S_k(z)$, creating three dimensional flow in the vicinity of the pumping well. This three dimensional flow exists even when wellbore storage is absent in the pumping well, which is different from the fact that uniform skin effect can influence aquifer drawdown only when wellbore storage exists in the pumping well. However, the three dimensional flow evolves to radial flow at farther distances, where the non-uniform skin effect is transformed into a uniform one that can be represented by a constant skin factor equal to the vertical average of $S_k(z)$ weighted by $Q_w(z, t)$. The conventional well hydraulics models of a constant skin factor can thus be used to deal with non-uniform skin problems in the radial flow regime or when vertically average drawdown is concerned.

Keywords: Non-uniform skin; Well hydraulics; Wellbore storage; Analytical solutions

1. Introduction

In well construction, drilling mud is usually employed for lubricating and cooling the drill bit, removing the drill cuttings, and stabilizing the borehole. Mud penetration may alter permeability of the porous formation surrounding the well screen, thereby creating a skin region around the well. The skin region having permeability less than that of the formation is called a positive skin; the reverse is called a negative skin. Under constant rate pumping, Van Everdingen (1953); Hurst (1953) assumed that a skin region of infinitesimal thickness results in a steady-state pressure drop across the wellbore face. This head discontinuity is characterized by a constant skin factor S_k. If there is no wellbore storage in the pumping well, S_k does not influence aquifer drawdown (Streltsova, 1988; Jargon, 1976; Chu et al., 1980; Moench, 1985; Kabala, 2001). Under constant

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Notations

$a_n(p)$	coefficients in (11)–(15), dimensionless.
$A_n(p)$	coefficients determined by the Gram–Schmidt method, dimensionless.
b	constant aquifer thickness, in [L].
$b_n(p)$	coefficients in (11)–(15), dimensionless.
C_D	coefficient of wellbore storage, dimensionless.
$F(\rho, p)$	radial component of $H_D(\rho,\zeta,p)$.
$G(\zeta, p)$	vertical component of $H_D(\rho,\zeta,p)$.
$h_D(\rho,\zeta,\tau)$	depth-specific drawdown, dimensionless.
$h_D^*(\rho,\tau)$	depth average drawdown, dimensionless.
$h_{wD}(\tau)$	drawdown in the pumping well, dimensionless.
$H_D(\rho,\zeta,p)$	Laplace-domain solution of $h(\rho,\zeta,\tau)$.
$H_D^*(\rho,p)$	Laplace-domain solution of $h^*(\rho,\tau)$.
$H_{wD}(p)$	Laplace-domain solution of $h_{wD}(\tau)$.
K_r	horizontal permeability of aquifer, in [L/T].
K_z	vertical permeability of aquifer, in [L/T].
$K_0(x)$	modified Bessel function of the second kind of order 0.
$K_1(x)$	modified Bessel function of the second kind of order 1.
p	Laplace transform parameter of τ.
Q	prescribed constant pumping rate, in [L^3/T].
$Q_w(\tau)$	total flow rate entering the well screen, dimensionless.
$Q_w(p)$	Laplace-domain solution of $Q_w(\tau)$.
$q_{wD}(\zeta,\tau)$	wellbore flux in the pumping well, dimensionless.
$q_{wD}(\zeta,p)$	Laplace-domain solution of $q_{wD}(\zeta,\tau)$.
r	radial distance from the pumping well, in [L].
r^*	extent of three-dimensional flow regime, [L].
r_c	casing radius of pumping well, in [L].
r_s	radius of skin region, in [L].
r_w	radius of the pumping well, in[L].
S	storage coefficient of aquifer, dimensionless.
S_k	conventional constant skin factor, dimensionless.
$\bar{S}_k$	equivalent constant skin factor of non-uniform skin, dimensionless.
$S_p(\rho,\zeta)$	defined by (26), dimensionless.
$S_k(\zeta)$	non-uniform skin function, dimensionless.
T	aquifer transmissivity, equal to $K_r b$, in [L^2/T].
$u_n(\zeta, p)$	non-orthogonal base functions, dimensionless.
$u_{mn}(p)$	defined by (18).
$w_n(p)$	defined by (17).
$Y(p)$	function to be approximated by the Gram–Schmidt method, dimensionless.
z	vertical distance, in [L].
α_i	positive skin factor of the ith section, dimensionless.
β	$(K_z/K_r)(r_w^2/b^2)$
χ_n	$(p+n^2\pi^2\beta)^{1/2}$.
γ	Euler's constant.
$\phi(\zeta)$	steady-state wellbore flux.
λ_n	defined by (27).
ρ	r/r_w.
r^*	r^*/r_w.
τ	Tt/Sr_w^2.
ζ	z/b.

head pumping, the use of S_k is also valid (Uraiet and Raghavan, 1980), yet S_k has influence on aquifer drawdown even if wellbore storage is absent in the pumping well (Chang and Chen, 2002; Chen and Chang, 2003). A negative skin cannot be characterized by changing S_k into $-S_k$ in the models because such a mathematical manipulation changes a prescribed pumping condition into an injection, or vice versa. Hurst et al. (1969) introduced the concept of effective well radius to deal with a negative skin.

Another way of dealing with a well skin is to assume that the skin region is a homogeneous porous annulus of finite extent embedded in an aquifer of differing properties (Clegg, 1967; Barker and Herbert, 1982; Faust and Mercer, 1984; Butler, 1988; Novakowski, 1989; Ruud and Kabala, 1997; Young, 1998; Peursema et al., 1999; Chen and Chang, 2002, and others). This approach, valid for both constant-rate and constant-head pumping, does not create a head discontinuity at the wellbore face and can be applied to positive or negative skin conditions.

However, certain complex skin conditions are beyond the description of these two approaches. For example, mud invasion, acidization processes, and stress redistribution may result in a skin region where permeability continuously varies as a function of the radial distance from the well (Bidaux and Tsang, 1991). In reservoir engineering, it is common to perforate the producing formation in several discrete intervals. Wells, either vertical or horizontal, completed in this way are referred to as 'selectively completed wells', of which each flow entry interval may be subject to a different skin effect (Yildiz and Cinar, 1998; Ozkan, 2001). Under field conditions, the fluid pressure, the concentration and particle sizes of the drilling mud, and porosity of the aquifer may all vary with depth. As a result, non-uniformly distributed mud penetration may create a skin region of varying thickness and/or permeability (Fig. 1). As opposed to the discrete non-uniform skin associated with a selectively completed well, Fig. 1 illustrates a continuous, or at least piece-wise continuous, non-uniform skin around a fully penetrating well. The purposes of this paper are (1) to evaluate the effect of non-uniform skin on aquifer response under constant rate pumping, and (2) to investigate the conditions under which its influence can be dealt with by a constant skin factor.

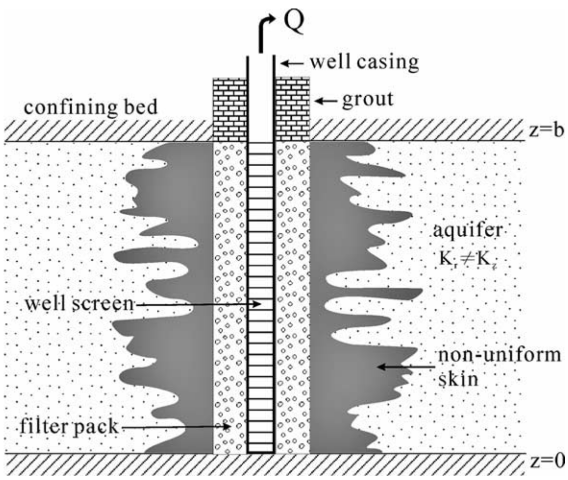

Fig. 1. Schematic diagram of possible non-uniform skin surrounding a well.

2. Model and solutions

2.1. Assumptions and model

A mathematical model based on Fig. 1 is developed here. In this model, a pumping well fully penetrates a homogeneous, anisotropic ($K_r \neq K_z$), confined aquifer of uniform thickness, b. A non-uniform skin region surrounds the screen section of the pumping well. As indicated by Hawkins (1956), the skin factor S_k can be defined in terms of aquifer and well characteristics as

$$S_k = \left(\frac{K}{K_s} - 1\right) \ln \frac{r_s}{r_w}, \quad S_k > 0 \tag{1}$$

where K and K_s are the hydraulic conductivity of the aquifer and the skin region, respectively; r_w and r_s are the radius of well and skin region, respectively. Eq. (1) indicates that S_k can be depth-dependent if K_s and/or r_s varies with depth. For the current study, therefore, the effect of a non-uniform skin is represented by a skin function $S_k(z)$ that is arbitrary and at least piecewise continuous over b. Without a loss of generality, the $S_k(z)$ of Fig. 2 is used for the evaluation of the non-uniform skin effect in the following discussion. It is a piecewise continuous function of NS sections. Each section is characterized by a constant positive skin factor, α_i such that an arbitrary skin function in the dimensionless form is

$$S_k(\zeta) = \alpha_i, \zeta_{i-1} \leq \zeta < \zeta_i, \ i = 1, 2, \cdots, NS$$
$$\zeta_0 = 0, \ \zeta_{NS} = 1 \tag{2}$$

where ζ is defined by z/b, and NS = 7 for the current study.

Pumping water from a well surrounded by a non-uniform skin induces horizontal (radial) and vertical flow in the aquifer, and hence the governing equation in dimensionless form is

$$\frac{\partial^2 h_D}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial h_D}{\partial \rho} + \beta \frac{\partial^2 h_D}{\partial \zeta^2} = \frac{\partial h_D}{\partial \tau} \tag{3}$$

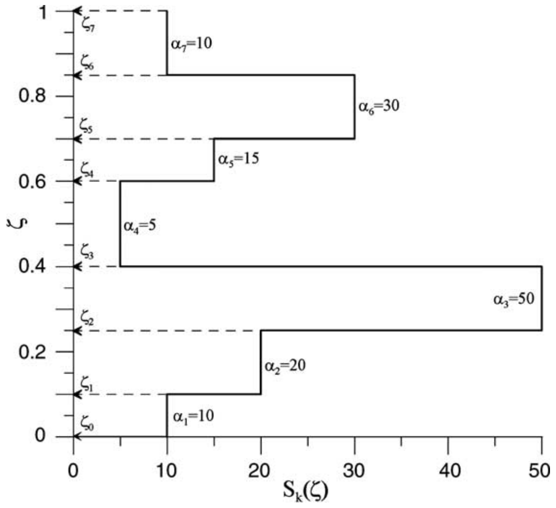

Fig. 2. An arbitrary piece-wise continuous skin function $S_k(\zeta)$ characterizes a possible non-uniform skin effect. Shown in the figure is the $S_k(\zeta)$ used to determine well bore flux and the head distribution in this study.

All the symbols are defined in Notation, unless otherwise noted. By invoking the infinite conductivity assumption, the head in the pumping well is independent of depth. It is assumed that wellbore storage occurs in the pumping well. As a result, the boundary conditions of the pumping well are

$$h_D(1,\zeta,\tau) - S_k(\zeta)\frac{\partial h_D}{\partial \rho}\Big|_{\rho=1} = h_{wD}(\tau) \tag{4}$$

$$-\int_0^1 \frac{\partial h_D}{\partial \rho}\Big|_{\rho=1} d\zeta + C_D\frac{dh_{wD}(\tau)}{d\tau} = 1 \tag{5}$$

where C_D denotes the constant wellbore storage coefficient. In (4), $\partial h_D/\partial_\rho$ at $\rho=1$ gives wellbore flux, $q_{wD}(\zeta,\tau)$, which is continuous across the infinitesimal skin thickness. The integral term in (5), therefore, defines the total flow rate entering the wellbore, $Q_w(\tau)$, which increases with time and tends to unity (the prescribed pumping rate Q in dimensionless form) as wellbore storage vanishes. Other necessary initial and boundary conditions are

$$h_D(\rho,\zeta,0) = h_{wD}(0) = 0 \tag{6}$$

$$h_D(\infty,\zeta,\tau) = 0 \tag{7}$$

$$\frac{\partial h_D}{\partial \zeta}\Big|_{\zeta=0,1} = 0 \tag{8}$$

Eqs. (3)–(8) together give a complete description of the problem of interest. If $S_k(\zeta)$ is set to a constant, there is no vertical flow component and the above model reduces to the radial flow model studied by Agarwal et al. (1970) who gave the solution of $h_{wD}(\tau)$, or by Chu et al. (1980) who presented the aquifer drawdown solution, $h_D^*(\rho,\tau)$.

2.2. Development of drawdown solution

The drawdown solution is determined first by applying to the model the Laplace transform with respect to time and the method of separation of variables. As discussed in detail in Appendix A, the Laplace domain solution of $h_D(\rho,\zeta,\tau)$ is

$$H_D(\rho,\zeta,p) = \sum_{n=0}^{\infty} A_n(p)K_0(\chi_n\rho)\cos(n\pi\zeta) \tag{9}$$

where $\chi_n=(p+n^2\pi^2\beta)^{1/2}$.The coefficients of $A_n(p)$, for $n=0,1,2,\ldots$, satisfy the following constraints

$$\sum_{n=0}^{\infty} A_n(p)u_n(\zeta,p) = Y(p), \quad 0\le\zeta\le 1 \tag{10}$$

where $Y(p)$ is $1/p$, and $u_n(\zeta, p)$ are

$$u_n(\zeta,p) = [a_n(p) + b_n(p)S_k(\zeta)]\cos(n\pi\zeta), \quad n = 0,1,2,\ldots \tag{11}$$

$$a_0(p) = \sqrt{p}K_1(\sqrt{p}) + pC_DK_0(\sqrt{p}) \tag{12}$$

$$b_0(p) = \sqrt[3]{pC_DK_1(\sqrt{p})} \tag{13}$$

$$a_n(p) = pC_DK_0(\chi_n), \quad n = 1,2,\ldots \tag{14}$$

$$b_n(p) = pC_D\chi_nK_0(\chi_n), \quad n = 1,2,\ldots \tag{15}$$

It is important to note that the infinite series of (9) is not a Fourier series because $u_n(\zeta, p)$ for $n=0,1,2,\ldots$ are not orthogonal in the interval $0\le\zeta\le 1$ due to the involvement of $S_k(\zeta)$. Hence, the determination of $A_n(p)$ requires an appropriate method such as the Gram–Schmidt method.

2.3. The Gram–Schmidt method

In essence, the Gram–Schmidt method is used to approximate a given function in terms of a set of orthogonal functions derived from a known set of non-orthogonal base functions. For the current study, the function to be approximated is $1/p$, which is denoted by $Y(p)$, and the base functions are $u_n(\zeta, p)$. In connection with the Gram–Schmidt method, (10) is rewritten as

$$\sum_{n=0}^{N} A_{N_n}(p)u_n(\zeta,p) \cong Y_N(p) \tag{16}$$

where N is a large integer, $A_{Nn}(p)$ denotes $A_n(p)$ for the N selected, and $Y_N(p)$ signifies the approximation of $Y(p)$ associated with $A_{Nn}(p)$. As N increases, $Y_N(p)$ approximates $Y(p)$ more closely, and as $N \to \infty$, $Y_N(p)$ becomes equal to $Y(p)$.

The evaluation of $A_{Nn}(p)$ in (16) can be made by the recursive relations developed by Kirkham and Powers (1972), where a number of parameters are calculated for each n, $n=0,1,2,\ldots,N$. Among these parameters, only two are directly dependent on the approximate function and the base functions, and they are defined by

$$w_n(p) = \int_0^1 Y(p)u_n(\zeta,p)\mathrm{d}\zeta, \quad n = 0,1,2,\ldots \tag{17}$$

$$u_{mn}(p) = \int_0^1 u_m(\zeta,p)u_n(\zeta,p)\mathrm{d}\zeta, \quad m = 0,1,2,\ldots,N; \quad n = 0,1,2,\ldots m \tag{18}$$

The other parameters in each iteration can be derived from $w_n(p)$ and $u_{mn}(p)$ using the formulae given by Kirkham and Powers (1972). After $A_{Nn}(p)$ are known, the Stehfest (1970) inversion method is applied to (9) to determine $h_D(\rho,\zeta,\tau)$.

Actually, the recursive relations are an explicit way of determining $A_{Nn}(p)$ for the following N's simultaneous linear algebraic equations

$$[u_{mn}]\{A_{Nn}\} = \{w_n\}, \quad m = n = 0,1,2,\ldots,N \tag{19}$$

where $[u_{mn}]$ is a $N \times N$ symmetrical matrix, $\{w_n\}$ and $\{A_{Nn}\}$ are $1 \times N$ column vectors. Alternatively, A_{Nn} can thus be determined by applying a standard matrix inversion method to (19); that is, $\{A_{Nn}\} = \{w_n\}[u_{mn}]^{-1}$, where $[u_{mn}]^{-1}$ is the inverse of $[u_{mn}]$. It has been verified that A_{Nn}'s determined by the recursive relations are nearly identical to those obtained using the matrix inversion method.

For the current study where (2) is used, closed-form expressions of $w_n(p)$ and $u_{mn}(p)$ can be obtained without difficulty by performing the integrations in (17) and (18). By the trial-and-error procedure, it is found that as N surpasses 1000 the approximation of (16) becomes less than 1% error.

2.4. Other relevant solutions

Aquifer drawdown from a fully penetrating observation well is regarded as vertical average drawdown, $h_D^*(\rho,\tau)$, which is the integration of $h_D(\rho,\zeta,\tau)$ with respect to ζ over (0, 1). Noting that the vertical integration of $\cos(n\pi\zeta)$ in (9) over (0, 1) is zero for $n \geq 1$ and is non-zero for $n=0$, the Laplace-domain solution of $h_D^*(\rho,\tau)$ is

$$H_D^*(\rho,p) = A_0(p)K_0(\rho\sqrt{p}) \tag{20}$$

The application of Darcy's law to (9) at $\rho=1$ yields the solution of wellbore flux in the Laplace domain

$$q_{wD}(\zeta,p) = -\left.\frac{\partial H_D}{\partial \rho}\right|_{\rho=1} = \sum_{n=0}^{\infty} A_n(p)\chi_n K_1(\chi_n)\cos(n\pi\zeta) \tag{21}$$

The integration of $q_{wD}(\zeta,p)$ with respect to ζ over (0, 1) gives the total flow rate in the Laplace domain

$$Q_w(p) = \int_0^1 q_{wD}(\zeta,p)d\zeta = A_0(p)\sqrt{p}K_1(\sqrt{p}) \tag{22}$$

The application of (9) to (A3) yields the Laplace domain solution of $h_{wD}(\tau)$

$$H_{wD}(p) = \frac{1}{C_D}\left[\frac{1}{p^2} - \frac{1}{\sqrt{p}}A_0(p)K_1(\sqrt{p})\right] \tag{23}$$

The Laplace inversion of (20)–(23) by the Stehfest method gives the respective solutions in the dimensionless time domain.

3. Analysis and discussion

3.1. Variation of wellbore flux

The distribution of $q_{wD}(\zeta,\tau)$ determined by (21) is shown in Fig. 3. The vertical variation of $q_{wD}(\zeta,\tau)$ at different times remains the same, and hence it can be represented by a time-independent function $\phi(\zeta)$, which in fact is the steady-state wellbore flux. Referring to Fig. 2, it is seen that $\phi(\zeta)$ is inversely related to $S_k(\zeta)$; that is, $\phi(\zeta)$ is largest in the interval of $0.4 \le \zeta \le 0.6$ where $S_k(\zeta)$ is smallest, and $\phi(\zeta)$ is smallest while $S_k(\zeta)$ is largest for $0.25 \le \zeta \le 0.4$. Dependent on the distribution of $S_k(\zeta)$, $\phi(\zeta)$ can be smaller or larger than unity, where the uniform wellbore flux is free from the non-uniform skin effect. However, $\phi(\zeta)$ shifts in the direction of increasing time until τ is greater than 10^6, responding to the temporal change of $Q_w(\tau)$. Therefore, wellbore storage ceases to influence wellbore flux at $\tau = 10^6$, and after then $Q_w(\tau)$ reaches its maximum of unity. As a result, $q_{wD}(\zeta,\tau)$ can be expressed as the product of $\phi(\zeta)$ and $Q_w(\tau)$. By this relation, it is understood that the integration of $\phi(\zeta)$ with respect to ζ over (0,1) must be equal to unity.

The influence of vertical anisotropy on $\phi(\zeta)$ is displayed in Fig. 4, where β is representative of the ratio of vertical anisotropy. While β is increased by 25 times from 4×10^{-6} to 10^{-4}, $\phi(\zeta)$ changes with a less than 6% maximum difference in the interval of $0.4 \le \zeta \le 0.6$. Thus, vertical anisotropy has little influence on wellbore flux distribution. However, this result only pertains to fully penetrating wells. For partially penetrating wells, the vertical anisotropy could influence the wellbore flux, as it would control the amount of flow to the pumping well from above or below the screened interval.

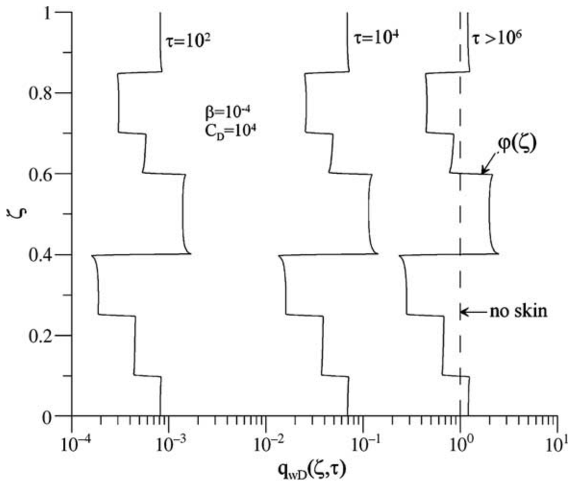

Fig. 3. The vertical variation of well bore flux, $q_{wD}(\zeta,\tau)$, at different dimensionless times remains the same and can be represented by the steady-state well bore flux that is denoted by $\phi(\zeta)$. $\phi(\zeta)$ is inversely related to $S_k(\zeta)$. The dashed line represents the uniform well bore flux which is free from the skin effect.

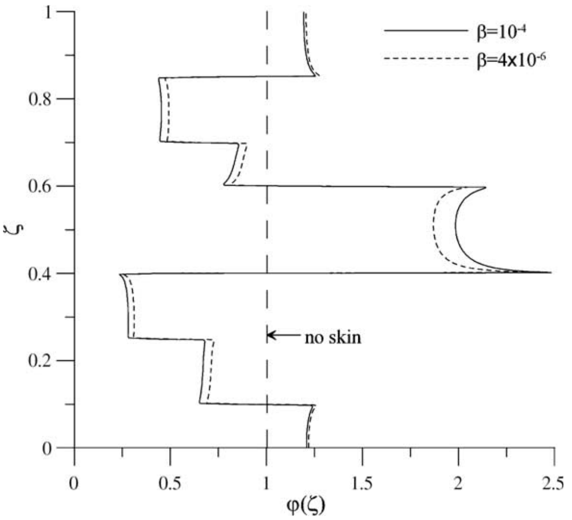

Fig. 4. The vertical anisotropy, represented by β, does not significantly influence $\phi(\zeta)$; a maximum 6% discrepancy occurs in $0.4 \le \zeta \le 0.6$, where $S_k(\zeta)$ is the smallest.

3.2. Variation of aquifer drawdown

The influence of $S_k(\zeta)$ on $h_D(\rho,\zeta,\tau)$ is shown in Fig. 5, in which three other solutions (i.e. vertical average drawdown by (20), the Theis solution, and the Chu et al. (1980) solution) are also included for comparison. At $\tau = 10^6$, three-dimensional (vertical and radially axi-symmetric) flow prevails for $1 \le \rho \le 50$, within which $h_D(\rho,\zeta,\tau)$ exhibits both radial and vertical variations. The vertical variations are in response to the distribution of $\phi(\zeta)$, as larger wellbore flux results in more drawdown and vice versa. In addition, the vertical average of $h_D(\rho,\zeta,\tau)$ coincides with the Chu et al. (1980) solution using a constant skin factor of 12.44. Therefore, the influence of $S_k(\zeta)$ on depth average drawdown can be lumped into an

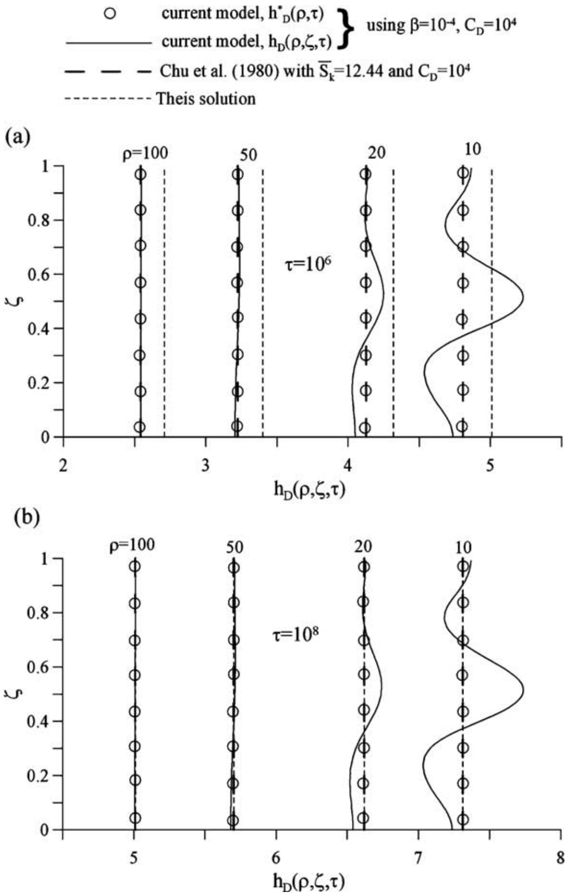

Fig. 5. Variations of $h_D(\rho,\zeta,\tau)$ at different ρ and ζ for (a) $\tau=10^6$ and (b) $\tau=10^8$. Three-dimensional flow prevails for $1\le\rho\le 50$, where $h_D(\rho,\zeta,\tau)$ is always under the influence of $S_k(\zeta)$. Three-dimensional flow changes to radial flow as $\rho>50$, where $h_D(\rho,\zeta,\tau)$ is independent of depth and can be determined by $h_D^*(\rho,\tau)$. $h_D^*(\rho,\tau)$ coincides with the Chu et al. (1980) solution using a constant skin factor of 12.44. $h_D^*(\rho,\tau)$ is less than at $\tau=10^6$ but equal to at $\tau=10^8$ the Theis solution, indicating that well bore storage vanishes at large times(e.g., $\tau>10^6$).

equivalent constant skin factor, $\bar{S}_k$, as discussed below. As distance increases, however, the vertical variations of $h_D(\rho,\zeta,\tau)$ diminish because vertical flow components at farther distances are smaller due to less influence by $S_k(\zeta)$. As ρ is greater than 100, vertical flow components become negligible and the three-dimensional flow evolves into a radial flow, where $h_D(\rho,\zeta,\tau)$ reduces to $h_D^*(\rho,\tau)$ of (20), which again can be matched by the Chu et al. (1980) solution of $\bar{S}_k=12.44$.

At $\tau=10^8$, the magnitude of $h_D(\rho,\zeta,\tau)$ or $h_D^*(\rho,\tau)$ is increased while the variations of $h_D(\rho,\zeta,\tau)$ or $h_D^*(\rho,\tau)$ are similar to those as occurred at $\tau=10^6$. But, now vertical average drawdown matches the Theis solution, indicating the absence of the compound effect of wellbore storage and skin. However, three-dimensional flow still takes place for $1\le\rho\le 50$, and thus aquifer drawdown in the neighborhood of the pumping well is under the influence of $S_k(\zeta)$, whether wellbore storage in the pumping well exists or not. The extent of the three-dimensional flow regime, ρ^*, can be approximated by

$$\rho^*\le 0.5/\sqrt{\beta} \text{ or } r^*\le 0.5b\sqrt{K_r/K_z} \tag{24}$$

where r^* is the dimensional ρ^*. A similar result exists for partial penetration effect. Hantush (1964) indicated that the three dimensional flow in association with a partially penetrating well changes to a radial type as $r\ge 1.5b\sqrt{K_r/K_z}$.

In summary, a non-uniform skin effect can be represented by a constant skin factor, provided aquifer drawdown is obtained from fully penetrating observation wells. And conventional well hydraulics theories of infinitesimal skin thickness approach are sufficient for data analysis. However, if drawdown measured by piezometers in the neighborhood of the pumping well varies with depth, the data analysis should be made with care concerning the possible influence of a non-uniform skin region around the pumping well.

On the other hand, the vertical anisotropy has significant influence on $h_D(\rho,\zeta,\tau)$ (see Fig. 6). Three values of β (i.e. 4×10^{-6}, 2.5×10^{-5}, and 10^{-4}) result in three distinctive drawdown curves with noticeable vertical variations that are inversely related to β. Therefore, the smaller the ratio of K_z/K_r the more pronounced is the $S_k(\zeta)$ effect on $h_D(\rho,\zeta,\tau)$.

3.3. Large-time approximation of $h_D(\rho,\zeta,\tau)$

As derived in Appendix B, the asymptotic solution of $h_D(\rho,\zeta,\tau)$ at large times is

$$h_D(\rho,\zeta,\tau)=\frac{1}{2}\ln[2.25\tau/\rho^2]+S_p(\rho,\zeta) \tag{25}$$

where

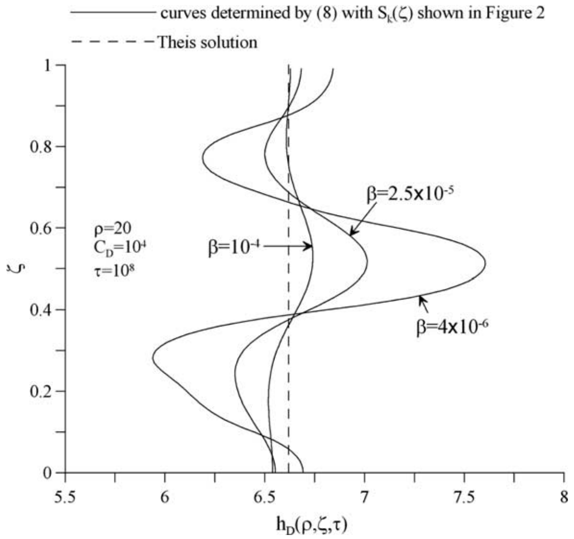

Fig. 6. In the three dimensional flow region, the vertical anisotropy significantly influences $h_D(\rho,\zeta,\tau)$.

$$S_p(\rho,\zeta) = 2\sum_{n=1}^{\infty} \frac{K_0(\rho n\pi\sqrt{\beta})}{K_1(n\pi\sqrt{\beta})}\frac{\lambda_n}{n\pi\sqrt{\beta}}\cos(n\pi\zeta) \qquad (26)$$

$$\lambda_n = \int_0^1 \varphi(\zeta)\cos(n\pi\zeta)\mathrm{d}\zeta, \quad n = 1,2,\ldots \qquad (27)$$

There is no C_D in (25) because wellbore storage is negligible at large times. The large-time asymptotic solution consists of a logarithmic approximation of the Theis solution, and a $S_k(\zeta)$-induced component. Accordingly, the semi-logarithmic plot of $h_D(\rho,\zeta,\tau)$ in Fig. 7 displays two parallel straight lines for large-time drawdown at $\zeta=0.3$ and 0.5, respectively. The vertical separation from the line of $\zeta=0.3$ or 0.5 to the line of the Theis approximation is $S_p(\rho,\zeta)$ of the respective depth. It is understood that $S_p(\rho,\zeta)$ of $\zeta=0.5$ is positive because $\phi(\zeta)$ of $\zeta=0.5$ is larger than unity, and $S_p(\rho,\zeta)$ of $\zeta=0.3$ is negative because $\phi(\zeta)$ of $\zeta=0.3$ is less than unity. Also, as ρ is large (e.g., $\rho \geq \rho^*$), the ratio of the Bessel functions in the infinite series of (26) approaches zero rapidly. Thus, aquifer drawdown at large times and far distances reduces to the Theis solution, as shown in Fig. 5b. If $\phi(\zeta)$ is set to unity as for uniform skin, λ_n is zero for $n \geq 1$, confirming that the uniform skin effect along with wellbore storage effect vanishes at large times.

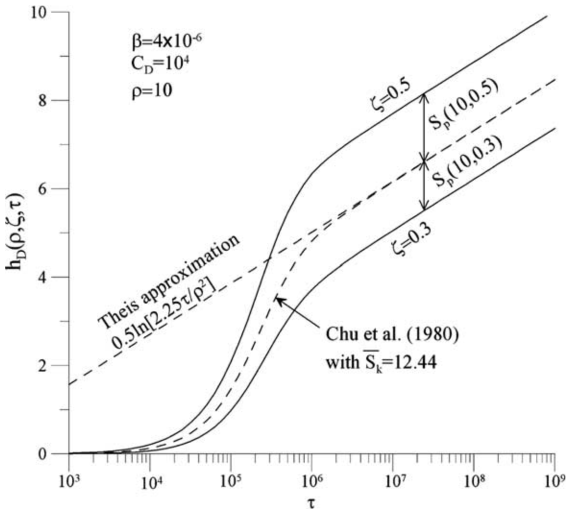

Fig. 7. Semi-logarithmic plot of $h_D(\rho,\zeta,\tau)$ of $\rho=10$ at $\zeta=0.3$ and 0.5. At large times, $h_D(\rho,\zeta,\tau)$ at different depths exhibits straight lines parallel to the Theis approximation. The vertical separation from the straight lines to the Theis approximation is $S_p(\rho,\zeta)$ of the respective depth.

3.4. Equivalent constant skin factor and associated solutions

To find an expression for the equivalent constant skin factor $\bar{S}_k$ in vertical average drawdown, (4) is integrated with respect to ζ over (0,1). Then

$$h_D^*(1,\tau) + \int_0^1 S_k(\zeta)\frac{\partial h_D}{\partial \rho}\Big|_{\rho=1}\mathrm{d}\zeta = h_{wD}(\tau) \qquad (28)$$

The gradient term inside the integral of (28) is the wellbore flux, which is the product of $Q_w(\tau)$ and $\phi(\zeta)$. The replacement of $Q_w(\tau)$ by an equivalent term ($\partial h_D^*/\partial\rho$ at $\rho=1$) enables (28) to be written as

$$h_D^*(1,\tau) - \bar{S}_k\frac{\partial h_D^*}{\partial \rho}\Big|_{\rho=1} = h_{wD}(\tau) \qquad (29)$$

which is the boundary condition subject to a constant skin factor in radial flow models, provided the skin factor is defined by

$$\bar{S}_k = \int_0^1 S_k(\zeta)\varphi(\zeta)\mathrm{d}\zeta \qquad (30)$$

For $S_k(\zeta)$ given in Fig. 2 and $\phi(\zeta)$ displayed in Fig. 3, $\bar{S}_k$ determined by (30) is 12.44, which has been used in previous discussion. Streltsova (1988) discussed the estimation of the constant skin factor using drawdown data measured at the pumping well during a constant-rate pumping test. This method is illustrated in Fig. 8, where $h_{wD}(\tau)$ was determined by (23) and $S_k(\zeta)$ given in Fig. 2. The drawdown points at large dimensionless times fall on a straight line of slope equal to 1.151. Extrapolation of this straight line to $\tau=1$ gives $h_{wD}(1)$, and then the constant skin factor is determined by (Streltsova, 1988; equation 2.66)

$$S_k = \frac{2\pi T}{Q} h_w(1) - \frac{1}{2}\ln\frac{2.25T}{Sr_w^2} \tag{30a}$$

of which the dimensionless form is

$$S_k = h_{wD}(1) - 0.405 \tag{30b}$$

Since $h_{wD}(1)$ is 12.845, S_k is 12.44, the same as that calculated by (30). Therefore, the constant skin factor as used in the infinitesimal thickness approach actually is the vertical average of the non-uniform distributed skin function weighted by wellbore flux as revealed in (30).

If $S_k(\zeta)$ in (11) is replaced by $\bar{S}_k$, $u_{mn}=0$ for $n\neq m$, and $u_{mn}\neq 0$ for $m=n$; that is, $u_n(\zeta, p)$ is orthogonal in

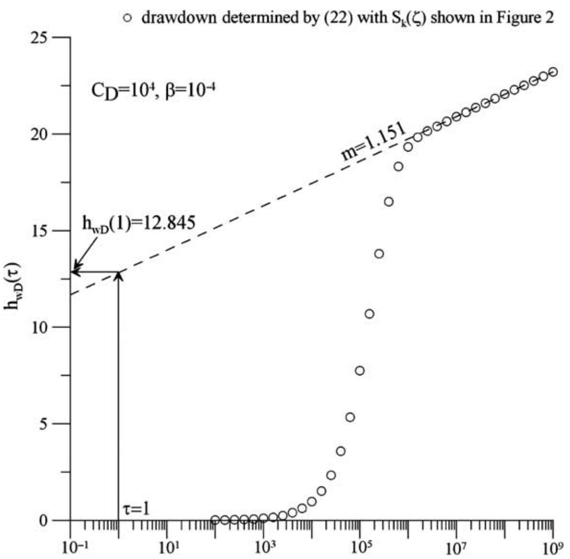

Fig. 8. At large times, semi-logarithmic plot of $h_{wD}(\tau)$ becomes a straight line, which is used in conjunction with (31), to estimate the equivalent constant skin factor, $\bar{S}_k$.

(0, 1). Then, in a straightforward manner, it can be found that A_n is zero for $n\geq 1$, and $A_0(p)$ is $[p(a_0+b_0\bar{S}_k)]^{-1}$, where $a_0(p)$ and $b_0(p)$ are defined by (12) and (13), respectively. As a result, (20) reduces to

$$H_D^*(\rho,p) = \frac{1}{p}\frac{K_0(\rho\sqrt{p})}{\sqrt{p}K_1(\sqrt{p}) + C_D p[K_0(\sqrt{p}) + S_k^*\sqrt{p}K_1(\sqrt{p})]} \tag{31}$$

which is identical to Eq. (9) of Chu et al. (1980), and (23) reduces to

$$H_{wD}(p) = \frac{1}{p}\frac{K_0(\sqrt{p}) + S_k^*\sqrt{p}K_1(\sqrt{p})}{\sqrt{p}K_1(\sqrt{p}) + C_D p[K_0(\sqrt{p}) + S_k^*\sqrt{p}K_1(\sqrt{p})]} \tag{32}$$

which is identical to Eq. (8) of Agarwal et al. (1970).

3.5. Absence of Wellbore storage

It is worthy mentioning that when wellbore storage is very small in the pumping well, the relevant solutions to the model cannot be derived from the above solutions by setting C_D to zero. This is because when C_D is zero (4) and (5) are no longer interrelated. Then, $A_0(p)$ alone can be determined by applying (9) to (A3). With this known $A_0(p)$, the application of (9) to (A2) results in a new set of $u_n(\zeta,p)$ for $n=1, 2,\ldots$, in (10), while $Y(p)$ remains as $1/p$. Again, the new base functions are non-orthogonal due to the involvement of $S_k(\zeta)$, and An(p) for $n=1, 2,\ldots$ can be determined by the Gram–Schmidt method.

4. Conclusion

1. The distribution of wellbore flux in the pumping well is inversely related to the variation of $S_k(z)$, inducing three dimensional flow in the neighborhood of the pumping well where aquifer drawdown changes in concert with $S_k(z)$. Regardless of whether wellbore storage exists in the pumping well or not, this three dimensional flow takes place for all times during the pumping period.
2. The three dimensional flow evolves to radial flow at far distance, where influence of $S_k(z)$ is

transformed into a uniform one that can be represented by a constant skin factor defined as the vertical average of $S_k(z)$ weighted by the wellbore flux of the pumping well. And its value can be estimated using the conventional well hydraulics method available for the determination of the skin factor. At late times after wellbore storage vanishes in the pumping well, aquifer drawdown in the radial flow regime is not influenced by the skin effect.

3. If aquifer drawdown measured by point-wise instrument in the neighborhood of the pumping well demonstrates depth dependence, the possible non-uniform skin effect should be taken into account in the data analysis.
4. If drawdown data are taken from fully penetrating observation wells or in the radial flow regime, the non-uniform skin effect can be studied using the conventional well hydraulics models of a constant skin factor.

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Appendix A. Determination of Laplace domain solution

Application of the Laplace transform with respect to τ to (3)–(8) yields

$$\frac{\partial^2 H_D}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial H_D}{\partial \rho} + \beta\frac{\partial^2 H_D}{\partial \zeta^2} - pH_D = 0 \tag{A1}$$

$$H_D(1,\zeta,p) - S_k(\zeta)\frac{\partial H_D}{\partial \rho}\Big|_{\rho=1} = H_{wD}(p) \tag{A2}$$

$$-\int_0^1 \frac{\partial H_D}{\partial \rho}\Big|_{\rho=1} d\zeta + pC_D H_{wD}(p) = \frac{1}{p} \tag{A3}$$

$$H_D(\infty,\zeta,\tau) = 0 \tag{A4}$$

$$\frac{\partial H_D}{\partial \zeta}\Big|_{\zeta=0} = 0 \quad \text{at } \zeta = 0 \text{ and } 1 \tag{A5}$$

By assuming that $H_D(\rho,\zeta,p)$ is the product of $F(\rho, p)$ and $G(\zeta, p)$, (A1) can be separated into the following two ordinary differential equations

$$\frac{d^2 G}{d\zeta^2} + \varepsilon^2 G = 0 \tag{A6}$$

$$\frac{d^2 F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} - \chi^2 F = 0 \tag{A7}$$

where $\chi = (p + \beta\varepsilon^2)^{1/2}$, and ε is constant. When $\varepsilon = n\pi$ for $n = 1,2\ldots$, there are infinite numbers of solutions that can satisfy (A6) subject to (A5), and they are

$$G(\zeta,p) = c_n(p)\cos(n\pi\zeta), \quad n = 0,1,2,\ldots \tag{A8}$$

where $c_n(p)$ are constant. Correspondingly, there are also infinite numbers of solutions to (A4) and (A7), such as

$$F(\rho,p) = d_n(p)K_0(\chi_n\rho), \quad n = 0,1,\ldots \tag{A9}$$

where $d_n(p)$ is constant and $\chi_n = (p + \beta n^2\pi^2)^{1/2}$. The linear combination of the products of (A8) and (A9) forms the complete solution of $H_D(\rho,\zeta,p)$

$$H_D(\rho,\zeta,p) = \sum_{n=0}^{\infty} A_n(p)K_0(\chi_n\rho)\cos(n\pi\zeta) \tag{A10}$$

where $A_n(p)$ is the product of $c_n(p)$ and $d_n(p)$. Substitution of (A2) into (A3) results in

$$-\int_0^1 \frac{\partial H_D}{\partial \rho}\Big|_{\rho=1} d\zeta + pC_D\Big[H_D(1,\zeta,p) - S_k(\zeta)\frac{\partial H_D}{\partial \rho}\Big|_{\rho=1}\Big] = \frac{1}{p} \tag{A11}$$

Substitution of (A10) into (A11) leads to (10).

Appendix B. Large-time approximation of $h_D(\rho,\zeta,\tau)$

In referring to (21), $q_{wD}(\zeta,p)$ can be expressed in terms of $\phi(\zeta)$ and $Q_w(p)$ such that

$$\sum_{n=0}^{\infty} A_n(p)\chi_n K_1(\chi_n)\cos(n\pi\zeta) = \varphi(\zeta)Q_w(p) \tag{B1}$$

Multiplying (B1) by $\cos(m\pi\zeta)$, where m is an integer, and then performing the integration with respect to ζ over (0,1), one obtains

$$\sum_{n=0}^{\infty} A_n(p)\chi_n K_1(\chi_n) \int_0^1 \cos(m\pi\zeta)\cos(n\pi\zeta)\mathrm{d}\zeta$$
$$= \lambda_m Q_w(p) \quad \text{(B2)}$$

where λ_m is defined by (27). The integral term in (B2) is unity for $m=n=0$, is 1/2 for $m=n\neq 0$, and is zero for $m\neq n$. As a result, $A_n(p)$ are expressed in terms of $Q_w(p)$ as

$$A_0(p) = \frac{Q_w(p)}{\sqrt{p}K_1(\sqrt{p})} \quad \text{(B3)}$$

$$A_n(p) = \frac{2\lambda_n Q_w(p)}{\chi_n K_1(\chi_n)} \quad n = 1, 2, 3, \ldots \quad \text{(B4)}$$

Substitution of (B3) and (B4) into (A10) gives an alternative form of (9)

$$H_D(\rho, \zeta, p) = Q_w(p)\left[\frac{K_0(\rho\sqrt{p})}{\sqrt{p}K_1(\sqrt{p})} + 2\sum_{n=1}^{\infty}\frac{K_0(\rho\chi_n)}{\chi_n K_1(\chi_n)}\lambda_n \cos(n\pi\zeta)\right] \quad \text{(B5)}$$

When p is small, $Q_w(p)$ can be approximated by $1/p$, χ_n by $n\pi\sqrt{\beta}$, $\sqrt{p}K_1(\sqrt{p})$ by unity, and $K_0(\rho\sqrt{p})$ by $-\ln(0.5\rho\sqrt{p}) - 0.5772$. As a result, (B5) of small p is

$$H_D(\rho,\zeta,p) = -\frac{1}{p}[\ln(0.5\rho\sqrt{p}) + \gamma] + \frac{1}{p}S_p(\rho,\zeta) \quad \text{(B6)}$$

Application of the Laplace inversion formula of Abramowitz and Stegun (1970; equation 29.3.98) to (B6) leads to (25).

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